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# Herbst inequalities for supercontractive semigroups

By

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## 1. Introduction

We will consider a probability space $(X, \mu)$ and a symmetric contraction semigroup $e^{-tA}$ on $L^2(X, \mu)$ which is Markovian in the sense that it takes nonnegative functions to nonnegative functions and satisfies $e^{-tA}1 = 1$ for all $t > 0$. For such a semigroup there is a rough equivalence between a knowledge of the norms, $N(t, p, q)$, of $e^{-tA}$ as an operator from $L^p(\mu)$ to $L^q(\mu)$ and a knowledge of the function $\beta$ in the family of logarithmic Sobolev inequalities

$$\int_X f(x)^2 \log \frac{|f(x)|}{\|f\|_2} d\mu(x) \leq \varepsilon \mathscr{E}(f, f) + \beta(\varepsilon)\|f\|_2^2, \quad \varepsilon > 0,\ f \in \mathscr{D}(A^{1/2}). \tag{1.1}$$

Here $\beta : (0, \infty) \to [0, \infty]$ may be taken to be a decreasing convex function. $\|f\|_2$ denotes the $L^2(\mu)$ norm and $\mathscr{E}(f, f) = \|A^{1/2}f\|_2^2$ is the Dirichlet form associated to $A$.

Although (1.1) will be assumed to hold throughout most of this paper for all $\varepsilon > 0$, the possibility that $\beta(\varepsilon) = \infty$ for some $\varepsilon$ means that (1.1) may have substance only for $\varepsilon$ in some interval $[\varepsilon_0, \infty)$ for some $\varepsilon_0 > 0$. For example if $\beta(\varepsilon) = \infty$ for $0 < \varepsilon < \varepsilon_0$ and $\beta(\varepsilon) = \beta_0$ for $\varepsilon \geq \varepsilon_0$ then (1.1) reduces to a standard logarithmic Sobolev inequality

$$\int_X f(x)^2 \log\left(\frac{|f(x)|}{\|f\|_2}\right) d\mu(x) \leq \varepsilon_0 \mathscr{E}(f, f) + \beta_0 \|f\|_2^2 \tag{1.2}$$

[G1], with fixed principal coefficient $\varepsilon_0$ and "local norm" $\beta_0$ (also known as the "defect").

Among the semigroups of interest to us there are three classes that have been distinguished up to now. Hypercontractive semigroups have the least smoothing ability: for $1 < p < q < \infty$ there is a minimum time $t_0 = t_0(p, q) > 0$ such that $\|e^{-tA}\|_{p \to q} < \infty$ if $t \geq t_0$. These semigroups are associated with the fixed logarithmic Sobolev inequality (1.2). On the other hnad if $\beta(\varepsilon) < \infty$ for all $\varepsilon > 0$ then $e^{-tA}$ is bounded from $L^p(\mu)$ to $L^q(\mu)$ for all $p$ and $q$ in $(1, \infty)$ and for all $t > 0$. Semigroups

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