ON THE SEMISIMPLICITY OF THE MODULAR REPRESENTATION ALGEBRA OF A FINITE GROUP

BY

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1. Introduction

1.1. Let G be a finite group with unit element e and K a field of prime characteristic p. By a G-module M we mean a (K, G)-module (elements of G act on the right). Denote by dim M the dimension of M as a K-module; we shall assume dim M is finite.

$\{M\}$ denotes the (K, G)-isomorphism class of M.

The modular representation algebra $A(K, G)$ is the linear algebra over the complex field $\boldsymbol{C}$ defined as follows:

The elements of $A(K, G)$ are the finite linear combinations over $\boldsymbol{C}$ of the G-module classes $\{M\}$, subject to the relations

$$\{M_1 + M_2\} = \{M_1\} + \{M_2\}$$

for all G-modules M_1, M_2. Here $M_1 + M_2$ denotes the direct sum $M_1 \oplus M_2$. Multiplication in $A(K, G)$ will be denoted by $\otimes$ and is defined by

$$\{M_1\} \otimes \{M_2\} = \{M_1 \otimes M_2\}$$

where $M_1 \otimes M_2$ is the tensor product over K, considered as a G-module by the rule $(m_1 \otimes m_2)x = m_1x \otimes m_2x$ $(m_1 \in M_1, m_2 \in M_2, x \in G)$.

By the Krull-Schmidt theorem for G-modules, $A(K, G)$ has as a basis (over $\boldsymbol{C}$) the classes of the indecomposable G-modules. By a theorem of D. G. Higman [5], the number of indecomposable classes is finite if and only if the Sylow p-subgroups of G are cyclic.

Let H be a subgroup of G. For any G-module M let M_H be the H-module formed by restriction of M to H; for any H-module L let L^G be the G-module induced from L. A G-module M is H-projective if there exists an H-module L such that M is isomorphic to a direct summand of L^G.

Denote by $A_H(K, G)$ the subspace of $A(K, G)$ spanned by the classes of H-projective G-modules. From the identity

$$L^G \otimes M \cong (L \otimes M_H)^G$$

which holds for any H-module L and G-module M it follows that $A_H(K, G)$ is an ideal of $A(K, G)$.

Let F be a subgroup of H. We shall write $F \leq H$. Let N be an F-module. Since $(N^H)^G \cong N^G$, a G-module which is F-projective is H-projective, i.e.

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