ON THE SPECTRAL RADIUS OF ELEMENTS IN A GROUP ALGEBRA[1]

BY

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Let G be a discrete group, $l_1(G)$ its group algebra. In [2], it was shown that if G contains a free non-abelian subsemigroup on two generators then $l_1(G)$ is non-symmetric. The proof, highly combinatorial in nature, rested on the fact that if x in $l_1(G)$ has no left inverse then there is a non-zero f in $l_1(G)^*$ such that $f(yx) = 0$ for each y in $l_1(G)$. This note contains the same theorem, but the proof given offers more insight into the group algebra.

Let $\mathcal{A}$ be a Banach *-algebra with identity e. $P(\mathcal{A})$ will denote the set of linear functionals on $\mathcal{A}$ such that $f(xx^*) \geq 0$ for each x in $\mathcal{A}$ and such that $f(e) = 1$. Implicit in the usual proof of Raikov's Theorem (c.f. Naimark [3]) is the following:

$\mathcal{A}$ is symmetric if, and only if, for each $x \in \mathcal{A}$,

$$\text{sp}\,(xx^*) \subset \{f(xx^*) \mid f \in P(\mathcal{A})\}.$$

(Symmetry is defined as in Rickart [4].) We require this result in proving

LEMMA 1. *Let $\mathcal{A}$ be a symmetric Banach *-algebra with identity. If x and y are normal elements oj $\mathcal{A}$ then*

$$\nu(xy) \leq \nu(x)\nu(y).$$

Proof. We first note that if z in $\mathcal{A}$ has no right inverse then there is a f in $P(\mathcal{A})$ such that $f(z) = 0$. Suppose z has no right inverse. Then zz^* has no right inverse, for if v were such an inverse then z^*x would be a right inverse for z. Hence

$$0 \in \text{sp}\,(zz^*).$$

By the preceding remark there is an $f \in P(\mathcal{A})$ such that $f(zz^*) = 0$. But then

$$|f(z)|^2 \leq f(zz^*)f(e) = 0,$$

and so

$$f(z) = 0.$$

A similar statement holds if z has no left inverse.

Suppose now that $\alpha \in \text{sp}\,(xy)$ and $|\alpha| = \nu(xy)$. Then $xy - \alpha e$ either has no left inverse or no right inverse. Hence there is an f_0 in $P(\mathcal{A})$ such that

$$f_0(xy - \alpha e) = f_0(xy) - \alpha = 0.$$

Hence

$$|f_0(xy)| = |\alpha| = \nu(xy).$$

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