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ON SETS CHARACTERIZING ADDITIVE AND MULTIPLICATIVE ARITHMETICAL FUNCTIONS

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1. Introduction

A function f: $\mathbf{N} \to \mathbf{C}$ is called *additive* if

$$f(mn) = f(m) + f(n) \tag{1}$$

for all coprime $m, n \in \mathbf{N}$. If (1) holds for all pairs of integers $m, n \in \mathbf{N}$, we say that f is *completely additive*. A function g: $\mathbf{N} \to \mathbf{C}$ is called *multiplicative* (resp. *completely multiplicative*) if

$$g(mn) = g(m)g(n) \tag{1'}$$

for all coprime $m, n \in \mathbf{N}$ (resp. for all $m, n \in \mathbf{N}$).

Because of the canonical representation

$$n = \prod_{p \text{ prime}} p^{\alpha_p} \quad \text{with} \quad p^{\alpha_p} \| n \tag{2}$$

of the integers $n \in \mathbf{N}$ we have $f(n) = \sum_{p \text{ prime}} f(p^{\alpha_p})$ (resp. $g(n) = \prod_{p \text{ prime}} g(p^{\alpha_p})$). An additive f can be extended uniquely to an "additive" function f^*: $\mathbf{Q}^+ \to \mathbf{C}$, where $\mathbf{Q}^+ = \{a/b: (a, b) = 1;\ a, b \in \mathbf{N}\}$, by $f^*(a/b) = f(a) - f(b)$. In a similar manner we get an extension g^* of a multiplicative function g by $g^*(a/b) = g(a)/g(b)$ in case $g(b) \neq 0$ for all $b \in \mathbf{N}$. In the following we denote by $\mathfrak{A}$ the set of all additive f: $\mathbf{Q}^+ \to \mathbf{C}$ and by $\mathfrak{M}$ the set of all multiplicative g: $\mathbf{Q}^+ \to \mathbf{C}$ with $g(b) \neq 0$ for all $b \in \mathbf{N}$. We write $\mathfrak{A}_c$ (resp. $\mathfrak{M}_c$) for the subsets of completely additive (resp. completely multiplicative) functions in $\mathfrak{A}$ (resp. $\mathfrak{M}$).

DEFINITIONS. Let $\mathscr{A} = \{a_n\} \subset \mathbf{Q}^+$. We say that $\mathscr{A}$ is a

(a) *U-set* for $\mathfrak{A}$ in case $f \in \mathfrak{A}$, $f(\mathscr{A}) = \{0\}$ implies $f = 0$,
(b) *U-set* for $\mathfrak{M}$ in case $g \in \mathfrak{M}$, $g(\mathscr{A}) = \{1\}$ implies $g = 1$,
(c) *C-set* for $\mathfrak{A}$ in case $f \in \mathfrak{A}$, $\lim_{n\to\infty} f(a_n) = 0$ implies $f = 0$,
(d) *C-set* for $\mathfrak{M}$ in case $g \in \mathfrak{M}$, $\lim_{n\to\infty} g(a_n) = 1$ implies $g = 1$.

In an obvious manner U-sets and C-sets are defined for $\mathfrak{A}_c$ (resp. $\mathfrak{M}_c$).

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