ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 1, Spring 1995

A GEOMETRIC HEAT FLOW FOR ONE-FORMS ON THREE DIMENSIONAL MANIFOLDS

STEVEN J. ALTSCHULER

1. Introduction

In this paper, we introduce a geometrically motivated heat flow for one-forms on 3-manifolds. Throughout, $(\mathbf{M}^3, g)$ is a Riemannian, compact, orientable 3-manifold and

$$\Omega^1_s(\mathbf{M}^3) \stackrel{\text{def}}{=} \left\{\alpha \in \Omega^1(\mathbf{M}^3) \big|\, |\alpha|^2 = 1\right\}.$$

In §3 we prove:

1.1. THEOREM. *Let $\alpha \in \Omega^1_s(\mathbf{M}^3)$, $\beta \in \Omega^1(\mathbf{M}^3) \times \mathbf{R}^+$. The weakly parabolic system*

$$\frac{\partial}{\partial t}\beta = *(\alpha \wedge df); \quad f \stackrel{\text{def}}{=} *(\alpha \wedge d\beta + \beta \wedge d\alpha),$$
$$\beta(\cdot, 0) = \alpha(\cdot) \tag{1.1}$$

has a unique, smooth solution for $t \in [0, \infty)$.

The evolution for the function f is also weakly parabolic and has the form

$$\frac{\partial f}{\partial t} = \Delta_\alpha f + \nabla_X f \tag{1.2}$$

where Δ_α is essentially the Laplacian on the null space of α and $X \in \mathfrak{X}(\mathbf{M}^3)$ is a smooth, time independent vector field. Let $d_\alpha(p, q)$ be the distance between $p, q \in \mathbf{M}^3$ restricted to the null space of α (see (2.1)). In §4 we prove a version of the strong maximum principle:

1.2. THEOREM. *Let f be a solution to (1.2) on $\mathbf{M}^3 \times [0, T]$. If $f(\cdot, 0) \geq 0$ and if $\exists q \in \mathbf{M}^3$ such that $f(q, 0) > 0$, then $f(p, t) > 0$ for all $t \in (0, T]$ and for all p such that $d_\alpha(p, q) < \infty$.*

Received October 2, 1992
1991 Mathematics Subject Classification. Primary 53C15; Secondary 53C12, 58G11, 57R30, 53C1S.