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A CONVERSE OF THE JORDAN-BROUWER THEOREM FOR QUASI-REGULAR IMMERSIONS

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1. Introduction

Suppose that $f\colon S^{n-1} \to S^n$ is a topological embedding. Then it is known as the Jordan-Brouwer Theorem that $f(S^{n-1})$ separates S^n into exactly two connected components. In [BR], C^1-immersions with normal crossings were studied and the following converse of the Jordan-Brouwer Theorem was obtained: if $f\colon S^{n-1} \to S^n$ is a C^1-immersion with normal crossings, then f is an embedding if and only if $f(S^{n-1})$ separates S^n into exactly two connected components. After that, this theorem has been generalized in various settings ([BMS1], [BMS2], [S]); however almost all of them have been involved with immersions *with normal crossings*.

The purpose of this paper is to consider a more general class of immersions than that of immersions with normal crossings, namely the class of quasi-regular immersions [H], and to obtain the converse of the Jordan-Brouwer Theorem. Recall that a C^1-immersion $f\colon M \to N$ into an n-dimensional manifold N is *quasi-regular* if the self-intersection locus $B \subset f(M)$ is an immersed submanifold of N with the property that for each $x \in B$ there is a coordinate system for N valid in a neighborhood U of x so that x corresponds to $0 \in \mathbf{R}^n$ and that the branches of f in U correspond to distinct linear subspaces of $\mathbf{R}^n$; i.e., given a numbering $y_1, y_2, \ldots, y_m$ of the points of $f^{-1}(x)$ there are pairwise disjoint neighborhoods $V_i \subset M$ around y_i so that $U \cap f(M) = U \cap (\cup_{i=1}^m f(V_i))$ is a union of m distinct linear subspaces of $\mathbf{R}^n$. It is clear that an immersion with normal crossings is always quasi-regular.

Our main result of this paper is the following.

THEOREM 1.1. *Let $f\colon M \to N$ be a quasi-regular immersion, where M is a closed connected $(n-1)$-dimensional manifold and N is a connected n-dimensional manifold. Assume that $H_1(M; \mathbf{Z}_2) = 0$ and $H_1(N; \mathbf{Z}) = 0$. Then if f is not an embedding, then $\beta_0(N - f(M)) \geq 3$, where β_0 denotes the number of connected components.*

Note that it has already been known that a proper codimension-1 quasi-regular immersion $f\colon M \to N$ separates N if $H_1(N; \mathbf{Z}_2) = 0$ [NR]. In fact, the same is true for proper C^1-immersions (see [HP], [F]).

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