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ON THE η-INVARIANT OF GENERALIZED ATIYAH-PATODI-SINGER BOUNDARY VALUE PROBLEMS

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1. Introduction. η-invariants for Dirac operators on manifolds with boundary

We consider a compact Riemannian manifold M with boundary N, $\dim M = 2k+1$ odd. Moreover let (S, ∇) be a complex Dirac bundle over M (cf. [17, Def. II.5.2]). Then we can form the Dirac operator

$$D\colon\ C_0^\infty(S) \to C_0^\infty(S)$$

associated to this structure. In order to obtain self-adjoint extensions of D we have to impose boundary conditions. We assume that the metric is product near the boundary, i.e., there is a collar $U = [0, 1) \times N$ of the boundary where the metric and the hermitian structure of S are product. Then on U the operator D has the form

$$D = \Gamma\left(\frac{\partial}{\partial x} + A\right), \tag{1.1}$$

where $\Gamma\colon\ S|N \to S|N$ is a unitary bundle automorphism (Clifford multiplication by the inward normal vector) and $A\colon\ C_0^\infty(S|N) \to C_0^\infty(S|N)$ is the corresponding Dirac operator on N. One easily checks the following identities

$$\Gamma^2 = -I,\ \Gamma^* = -\Gamma,\ \Gamma A = -A\Gamma,\ A^* = A. \tag{1.2}$$

In order to define self-adjoint boundary conditions for D we first deal with the case $\ker A = \{0\}$, i.e., A is invertible. This case is most similar to [1] and there is a canonical self-adjoint boundary condition. Let $\Pi_\pm$ be the orthogonal projection onto the positive (negative) spectral subspace of A, i.e. $\Pi_+ = 1_{(0,\infty)}(A)$, $\Pi_- = 1_{(-\infty,0)}(A)$. We use the pseudodifferential operator Π_+ as elliptic boundary condition and put

$$\begin{aligned} D_+ &:= D, \\ \mathcal{D}(D_+) &:= \{s \in H^1(M, S) \mid \Pi_+(s|N) = 0\}. \end{aligned} \tag{1.3}$$

where H^k denotes the k-th Sobolev space and $\mathcal{D}(\cdot)$ denotes the domain of an operator. The elliptic boundary conditions for Dirac operators have been discussed in [3],

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