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RESTRICTION THEOREMS RELATED TO ATOMS

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Introduction

Let $\mathbb{R}^n$ be n-dimensional real Euclidean space and let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Suppose that $d\sigma = d\sigma(x')$ is the element of Lebesgue measure on S^{n-1} so that the measure of S^{n-1} is 1. If $d\mu = \psi d\sigma$ is a measure with smooth density ψ, then from [9] or [10] we know that the Fourier transform of $d\mu$ satisfies $d\hat{\mu}(\xi) = O(|\xi|^{-\varepsilon})$ as $|\xi| \to \infty$, for some $\varepsilon > 0$. It turns out that if the density ψ is merely in $L^p(d\sigma)$, for some $p > 1$, then there is still an average decrease of $d\hat{\mu}$ at infinity along any ray emanating from the origin. More precisely, suppose that ψ is in $L^p(d\sigma)$, then

$$(*) \qquad R^{-1}\int_O^R |d\hat{\mu}(\rho\xi)|^2 d\rho \leq A(R|\xi|)^{-\varepsilon},$$

where $\varepsilon < (1-p^{-1})/2$, and A is a positive constant independent of $R|\xi|$ (see [10]). The estimate $(*)$ has the following application.

Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$, with $\Omega \in L^p(S^{n-1})$, for some $p > 1$, and $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Let $r \to b(r)$ be a bounded function on $(0,\infty)$. We consider the distribution $K = \text{P.V.}b(|x|)\Omega(x)|x|^{-n}$ and study the boundedness of the operator Tf which is defined by $Tf = f * K$. This operator was studied extensively and its boundedness properties were established in R. Fefferman [7], Namazi [8], Duoandikoetxea and Rubio de Francia [4] and Chen [1]. In his new significant book [9], by using $(*)$, E. M. Stein gives an alternative proof to conclude that, under the restriction $n \geq 2$, the mapping $f \to f * K$ extends to a bounded operator in $L^2(\mathbb{R}^n)$. Meanwhile, he points out that the condition $b \in L^\infty(0,\infty)$ can be replaced by a weaker condition (see pages 372–373 in [10]; also see [4]):

$$(1) \qquad R^{-1}\int_O^R |b(\rho)|^2 d\rho \leq A \text{ for all } R > 0.$$

In this paper, we shall study $d\mu = \psi d\sigma$ where the density ψ is an atom. As an application, we will prove that if $\Omega(x')$ is merely in the Hardy space $H^1(S^{n-1})$ with mean zero property and if, for some $p > 1$, the radial function $b(|x|)$ satisfies

$$(1') \qquad R^{-1}\int_O^R |b(\rho)|^p d\rho \leq A \text{ for all } R > O,$$

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