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ON THE BERGMAN INVARIANT AND CURVATURES OF THE BERGMAN METRIC

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Introduction

Let Ω be a (bounded) domain in $\mathbb{C}^n$ and $A^2(\Omega)$ the Bergman space of holomorphic functions in $L^2(\Omega)$. Let $\{\varphi_n\}$ be an orthonormal basis for $A^2(\Omega)$ with respect to the L^2-inner product. The Bergman kernel K_Ω associated to Ω is given by

$$K_\Omega(z) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(z)}, \qquad \forall z \in \Omega.$$

This kernel, which is independent of the choice of orthonormal basis (see [KR2]), gives rise to an invariant metric, the Bergman metric, on Ω as follows:

$$B_\Omega(z, u) \equiv \left(\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar\beta}(z) u_\alpha \overline{u}_\beta \right)^{1/2} \quad \text{where} \quad g_{\alpha\bar\beta}(z) = \frac{\partial^2 \log K_\Omega(z)}{\partial z_\alpha \partial \bar z_\beta}.$$

Then the *Bergman canonical invariant* is defined by

$$J_\Omega(z) \equiv \frac{\det G_\Omega(z)}{K_\Omega(z)} \quad \text{where} \quad G_\Omega(z) = \left(g_{\alpha\bar\beta}(z)\right).$$

Associated with the volume form of the Kähler metric B_Ω is the Ricci curvature (tensor) $R_\Omega = \sum R_{\alpha\bar\beta} dz_\alpha d\bar z_\beta$ which is given by

$$R_{\alpha\bar\beta} = -\frac{\partial^2 \log \det G_\Omega}{\partial z_\alpha \partial \bar z_\beta}.$$

It is easy to see that both J_Ω and R_Ω are invariant under biholomorphic mappings. There is an intrinsic relation between these two invariants; details are provided in Proposition 2.1.

The invariant J_Ω was first introduced by Bergman [BER] to define a general Kähler metric. It is also very useful in deriving various distortion theorems ([BER], [STA]) and in characterizing *representative domains* [SPR]. The interest in the invariant J_Ω

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