GENERALIZATIONS OF SOME COMBINATORIAL INEQUALITIES OF H. J. RYSER[1]

BY

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I. Introduction and results

In a recent interesting paper, H. J. Ryser obtained the following results [1].

Let H be a nonnegative hermitian matrix of rank e and order v with eigenvalues $\lambda_1, \cdots, \lambda_v$, where $\lambda_1 \geqq \cdots \geqq \lambda_e > \lambda_{e+1} = \cdots = \lambda_v = 0$. Let h be an integer, $h > 1$, such that $e \leqq h \leqq v$, and define k and λ by

$$\text{trace } (H) = kh, \qquad \lambda_h \leqq k + (h-1)\lambda \leqq \lambda_1 .$$

Define the matrix B of order h by

$$B = (k - \lambda)I + \lambda J,$$

where I is the identity matrix and J is the matrix all of whose entries are 1's. Let

$$B_0 = B \dotplus 0,$$

where the matrix B_0 of order v is the direct sum of the matrix B of order h and the zero matrix of order $(v - h)$. Let

$$k^* = \text{trace } (H)/v, \qquad \mu = \textstyle\sum_{i=1}^{v} \sum_{j=1}^{v} h_{ij}, \qquad \lambda^* = ((\mu/v) - k^*)/(v-1).$$

Define the matrix B^* of order v by

$$B^* = (k^* - \lambda^*)I + \lambda^* J.$$

Finally let $C_r(A)$ denote the r^{th} compound matrix of A, and let $P_r(A)$ denote the r^{th} induced power matrix of A (for definitions of C_r and P_r see [1]). Then we have

Theorem 1. *The matrices H and B_0 satisfy*

$$\text{trace } (C_r(H)) \leqq \text{trace } (C_r(B_0)) \qquad (1 \leqq r \leqq v).$$

Equality holds for $r = 1, h + 1, \cdots, v$. If $k + (h - 1)\lambda \neq 0$ and equality holds for an r, $1 < r \leqq h$, or $k + (h - 1)\lambda = 0$ and equality holds for an r, $1 < r < h$, then there exists a unitary U such that $H = U^{-1}B_0 U$.

Theorem 2. *The matrices H and B^* satisfy*

$$\text{trace } (C_r(H)) \leqq \text{trace } (C_r(B^*)) \qquad (1 \leqq r \leqq v).$$

Received May 18, 1962.

[1] This research was supported by the Air Force Office of Scientific Research.