ON THE CATEGORY OF ERGODIC MEASURES

BY

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Introduction

If (X, S, μ) is a finite measure space and G the group of all one-to-one measure-preserving transformations, then two interesting topologies can be assigned to G which make it a topological group. In certain dynamical problems it is of interest to know whether a particular transformation is ergodic or not. Even though this problem has not been solved till now, the existence of a large class of ergodic transformations has been shown by the determination of their category in G. In particular, when the measure space is nonatomic, Halmos [1] proved that the set of weakly mixing transformations is a dense G_δ in G under the weak topology. Similar results were proved by Oxtoby and Ulam [2]. Rokhlin [3] proved that under the same weak topology in G, the set of strongly mixing transformations is a set of the first category.

In problems of information-theoretic interest, we have a measurable space $(X, \mathcal{S})$ and a one-to-one both ways measurable map T of X onto itself. Here, it is of interest to know whether there are a lot of ergodic measures in the space of invariant measures. In order to study this problem, we take X to be a topological space, $\mathcal{S}$ the Borel σ-field, and T a homeomorphism of X onto itself. Then several topologies can be assigned to the space of invariant probability measures. Taking X to be a complete and separable metric space and assigning the weak topology to the space of invariant probability measures, we show that the set of ergodic measures is a G_δ. When X is a countable product of complete and separable metric spaces and T is the shift transformation, we show that the ergodic measures form a dense G_δ under the same topology. Examples are given to show that the ergodic measures need not be dense in the general case. In the case of the shift transformation we have proved that the set of strongly mixing measures is a set of the first category.

1. Preliminaries

Let $(X, \mathcal{S})$ be any measurable space, and T a one-to-one both ways measurable map of X onto itself. Whenever the space X is a topological space, we take $\mathcal{S}$ to be the Borel σ-field, and T a homeomorphism of X onto itself. By a measure, we always mean a probability measure. We denote by $\mathfrak{M}$, $\mathfrak{M}_e$, and $\mathfrak{M}_s$ the space of all invariant, ergodic, and strongly mixing measures, respectively. For these definitions we refer to [1].

A point $x \in X$ will be called periodic if for some integer k, $T^k x = x$. A measure $\mu \in \mathfrak{M}$ is periodic if for some integer k, $\mu(A \cap T^k A) = \mu(A)$ for all

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