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RIGIDITY RESULTS FOR GROUPS WITH RADICAL COHOMOLOGY OF FINITE GROUPS AND ARITHMETICITY PROBLEMS

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1. Introduction. A *linear algebraic group G defined over* $\mathbb{Q}$ is a subgroup $G \leq \mathbf{GL}(n, \mathbb{C})$ $(n \in \mathbb{N})$ that is also an affine algebraic set defined by polynomials with rational coefficients in the natural coordinates of $\mathbf{GL}(n, \mathbb{C})$. (We also say "$\mathbb{Q}$-defined" to mean "defined over $\mathbb{Q}$".) If R is a subring of $\mathbb{C}$, we then put $G(R) := G \cap \mathbf{GL}(n, R)$. Of course, we have $G = G(\mathbb{C})$. Let G be a linear algebraic group defined over $\mathbb{Q}$. A subgroup $\Gamma \leq G$ is called an *arithmetic subgroup* of G if Γ is commensurable with $G(\mathbb{Z})$. An abstract group Δ is called arithmetic if it is isomorphic to an arithmetic subgroup of a $\mathbb{Q}$-defined linear algebraic group. Two subgroups Γ_1, Γ_2 of $\mathbf{GL}(n, \mathbb{C})$ are called *commensurable* if their intersection $\Gamma_1 \cap \Gamma_2$ has finite index both in Γ_1 and Γ_2.

The main aim of this work is concerned with the structure of finite extension groups of arithmetic groups. We show by means of examples that these need not be arithmetic groups, contrary to the general expectation. We are also able to give a strong criterion for them to have this property. We reach this aim by an application of new rigidity results for general arithmetic groups. Let G_1, G_2 be two linear algebraic groups defined over $\mathbb{Q}$ and $\Gamma_1 \leq G_1$, $\Gamma_2 \leq G_2$ arithmetic subgroups. Rigidity problems are concerned with the question of whether an isomorphism between the arithmetic groups Γ_1, Γ_2 can be extended to a rational isomorphism between the algebraic groups G_1, G_2. For $\mathbb{Q}$-simple linear algebraic groups G that are defined over $\mathbb{Q}$ and of adjoint type, the results of Margulis [Ma], Mostow [Mo], and Prasad [Pr1] establish a positive answer to the rigidity problems under a condition on the real ranks of G_1, G_2. We analyze here the general case in which G_1, G_2 might have a

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