THE DISTRIBUTION OF SPACINGS BETWEEN QUADRATIC RESIDUES

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1. Introduction. Our goal in this paper is to study the distribution of spacings (or gaps) between squares in $\mathbf{Z}/q\mathbf{Z}$, as $q \to \infty$. In the case that q is prime, a theorem of Davenport (see [3], [4], [11], and [18]) shows that the probability of two consecutive quadratic residues modulo a prime q being spaced h units apart is 2^{-h}, as $q \to \infty$. For our purposes, we may interpret this result as saying that when we normalize the spacings to have unit mean, then the distribution of spacing as $q \to \infty$ along primes is given by

$$P(s) = \sum_{h=1}^{\infty} 2^{-h} \delta\left(s - \frac{h}{2}\right),$$

that is, a sum of point masses at half-integers with exponentially decreasing weights.

In this paper, we study the spacing distribution of squares modulo q when q is square-free and *highly composite,* that is, the limiting distribution of spacings between the squares modulo q as the number of prime divisors, $\omega(q)$, tends to infinity. For odd square-free q, the number N_q of squares modulo q equals

$$N_q = \prod_{p|q} \frac{p+1}{2}.$$

This is because, if p is an odd prime, the number of squares modulo p is $(p+1)/2$ and, for q square-free, x is a square modulo q if and only if x is a square modulo p for all primes p dividing q. Thus, for odd q, the mean spacing $s_q = q/N$ equals

$$s_q = \frac{2^{\omega(q)}}{\prod_{p|q}(1+1/p)} = \frac{2^{\omega(q)}}{\sigma_{-1}(q)}.$$

For $q = 2q'$ even and square-free, it is easily seen that $s_q = s_{q'}$. It follows that $s_q \to \infty$ as $\omega(q) \to \infty$, unlike the case of prime q, where the mean spacing is essentially constant. Thus, unlike in the prime case (where the level spacing distribution was forced to be supported on a lattice), in the highly composite case, there is an a priori chance of getting a continuous distribution.

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