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SET-THEORETICAL SOLUTIONS TO THE QUANTUM YANG-BAXTER EQUATION

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0. Introduction. The quantum Yang-Baxter equation (QYBE) is one of the basic equations in mathematical physics that lies in the foundation of the theory of quantum groups. This equation involves a linear operator $R : V \otimes V \to V \otimes V$, where V is a vector space, and has the form

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \quad \text{in } \operatorname{End}(V \otimes V \otimes V),$$

where R^{ij} means R acting in the ith and jth components.

In the last fifteen years, many solutions of this equation were found, and the related algebraic structures (Hopf algebras) have been intensively studied. However, these solutions were usually "deformations" of the identity solution. On the other hand, it is interesting to study solutions that are not obtained in this way. In [Dr], Drinfeld suggested the study of the simplest class of such solutions, the so-called set-theoretical solutions. By definition, a set-theoretical solution is a solution for which V is a vector space spanned by a set X, and R is the linear operator induced by a mapping $X \times X \to X \times X$. One construction of set-theoretical solutions to the QYBE is given in [WX], and we recently became aware of a general construction of such solutions in [LYZ].

In this paper, we study set-theoretical solutions of the quantum Yang-Baxter equation, satisfying additional conditions: invertibility, unitarity, and nondegeneracy. They turn out to have many beautiful properties. We discuss the geometric and algebraic interpretations of such solutions, introduce several constructions of them, and give their classification in terms of group theory.

Section 1 contains the background material. In Section 1.1, we give the main definitions and the simplest examples. We introduce the notion of a nondegenerate symmetric set, which is a set X with an invertible mapping $R : X^2 \to X^2$ satisfying the quantum Yang-Baxter equation and the nondegeneracy and unitarity conditions. We explain that if X is a nondegenerate symmetric set, then the set X^n has a natural action of the symmetric group S_n, called the twisted action, which is, in general, different from the usual action by permutations. In Section 1.2, we show that any nondegenerate symmetric set defines a coloring rule for collections of closed smooth curves in the plane, under which the number of colorings depends only on the number

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