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THE NUMBER OF REPRESENTATIONS OF AN INTEGER BY A QUADRATIC FORM

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Introduction. Our quadratic form $x \mapsto \varphi[x]$ is defined on a vector space V of dimension n (> 1) over a totally real algebraic number field F. We let G^φ denote the orthogonal group of φ, and $\mathfrak{g}$ the maximal order of F. For a $\mathfrak{g}$-lattice L in V and an element $h \in \mathfrak{g}$, we denote by $N(L, h)$ the number of elements $x \in L$ such that $\varphi[x] = h$. Here we assume that φ is totally positive definite, h is totally positive, and $\varphi[x] \in \mathfrak{g}$ for every $x \in L$. As previous researchers on this topic discovered, in order to obtain a meaningful formula in the most general case, we have to consider a certain average of several numbers of type $N(L, h)$ instead of a single $N(L, h)$ as follows. We first take a set of lattices $\{L_i\}_{i=1}^k$ that are representatives for the classes belonging to the genus of L. Then we put

$$\mathfrak{m}(L) = \sum_{i=1}^{k} [\Gamma_i : 1]^{-1},$$

$$R(L, h) = \sum_{i=1}^{k} [\Gamma_i : 1]^{-1} N(L_i, h),$$

where $\Gamma_i = \{\gamma \in G^\varphi \mid L_i\gamma = L_i\}$. The purpose of this paper is to give an exact formula for $R(L, h)$ when L is *maximal* in the sense that it is maximal among the lattices on which φ takes values in $\mathfrak{g}$. In the previous work [S5], we gave an exact formula for $\mathfrak{m}(L)$. Thus the present paper is its natural continuation.

Before stating the formula, let us first recall the result of Siegel on this topic. For a prime ideal $\mathfrak{p}$ in F and $0 < m \in \mathbf{Z}$, let $A_m(\mathfrak{p})$ denote the number of elements y in $L/\mathfrak{p}^m L$ such that $\varphi[y] - h \in \mathfrak{p}^m$. Then, as Siegel showed, $N(\mathfrak{p})^{m(1-n)} A_m(\mathfrak{p})$ is a constant $d_\mathfrak{p}(h)$ independent of m if m is sufficiently large. Suppose that V (resp. L) is the vector space (resp. the module) of all n-dimensional row vectors with entries in F (resp. $\mathfrak{g}$) and $\varphi[x] = x\varphi_0 \cdot {}^t x$ with a totally positive symmetric matrix φ_0 with entries in $\mathfrak{g}$. Then he proved that

$$(1) \qquad \frac{R(L, h)}{\mathfrak{m}(L)} = c_n D_F^{(1-n)/2} \pi^{dn/2} \Gamma\left(\frac{n}{2}\right)^{-d} N_{F/\mathbf{Q}}\big(\det(\varphi_0)^{-1} h^{n-2}\big)^{1/2} \prod_{\mathfrak{p}} d_\mathfrak{p}(h),$$

where D_F is the discriminant of F, $\mathfrak{p}$ runs over all the prime ideals in F, $c_n = 1$ if $n > 2$, $c_n = 1/2$ if $n = 2$, and $d = [F : \mathbf{Q}]$; the infinite product $\prod_\mathfrak{p} d_\mathfrak{p}(h)$ must be

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