Vol. 99, No. 1 DUKE MATHEMATICAL JOURNAL © 1999

AMENABILITY, BILIPSCHITZ EQUIVALENCE, AND THE VON NEUMANN CONJECTURE

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1. Introduction and statement of results. The geometry of discrete metric spaces has recently been a very active research area. The motivation comes primarily from two (not completely separate) areas of mathematics. The first is the study of noncompact manifolds equipped with metrics well defined up to uniformly bounded distortion. The equivalence classes of these manifolds arise naturally in the study of compact manifolds—for example, as universal covers or leaves of foliations. To focus attention purely on the large-scale structure, the local topology is thrown away by passing to a discrete subset, called a net, which evenly fills out the space. A typical example is $\mathbb{Z}^n$ in $\mathbb{R}^n$. Two nets in a space are bilipschitz equivalent on the large scale, but not the small. This leads to the notion of a quasi-isometry of metric spaces and gives a functor from manifolds of bounded geometry, up to bounded distortion, to discrete metric spaces, up to quasi-isometry.

The second source of motivation and examples is geometric group theory. A finitely generated group can be given a metric by measuring the distance from an element to the identity by the minimal length of a word in the generators representing the element. This extends to a metric by left translation invariance. This metric, called the (left) word metric, is well defined up to bilipschitz equivalence. (This can be seen simply by expanding one set of generators in another.) One can then study the group via the geometry of this space.

The coarse geometry of manifolds and geometric group theory are related, most obviously by the fundamental group. If M is a compact manifold, then $\bar{M}$ has obvious nets given by the orbits of the action of $\pi_1(M)$. These nets are canonically identified, at least as sets, with $\pi_1(M)$. Conversely, if Γ is any finitely presented group, there is a compact manifold with $\pi_1 = \Gamma$. It is a fundamental observation in the subject that the quasi-isometry type of Γ as a net agrees with the word metric. More generally, the Milnor-Svarc theorem says that any path metric space on which Γ acts cocompactly by isometries is quasi-isometric to Γ with its word metric. Thus, for instance, any two compact hyperbolic manifolds of the same dimension have quasi-isometric fundamental groups. The same holds for other locally symmetric spaces. This is the sense in which we expect the word metric to capture the "geometric type" of a group.

The reader may have noticed a small distinction between the geometric and algebraic viewpoints: nets are well defined only up to quasi-isometry, while the word met-

Received 6 October 1997. Revision received 15 June 1998.
1991 *Mathematics Subject Classification.* Primary 20F32; Secondary 20E06, 05C25.