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THE RANK OF QUOTIENTS OF $J_0(N)$

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CONTENTS

1. Introduction.

Given a large prime N, consider the Jacobian variety $J_0(N)$ of the modular curve $X_0(N) = \mathbf{H}/\Gamma_0(N)$. There has been considerable study recently of the rank over $\mathbf{Q}$ of these varieties and their quotients, much of it dealing with two questions:

- Are there large quotients of $J_0(N)$ with rank zero over the rationals?
- How large a rank can a quotient of $J_0(N)$ have?

See Mazur [9] and Merel [10] for arithmetic approaches to the first question; one consequence of the latter is the existence of a quotient with rank zero and dimension at least $N^{1/8}$ (out of a maximum possible dimension of $D_N = N/12 + \mathcal{O}(1)$, since N is prime). The progress of Kolyvagin [8] and Gross and Zagier [6] toward the Birch-Swinnerton-Dyer conjecture allows one to investigate these questions through the "analytic rank" of $J_0(N)$. This is defined (see, for example, [1]) to be the order of the zero at $s = 1$ of $L(J_0/\mathbf{Q}, s) = \prod_f L_f(s)$, where the product is taken over L-functions associated with the Hecke eigenbasis of $S_2(\Gamma_0(N))$, the weight-2 holomorphic forms on $X_0(N)$ (of which there are D_N). In particular, if one can show that many of the $L_f(s)$'s have order zero or one, then the corresponding quotients provide good answers to each of the questions. Along these lines, Duke [4] has used first and second moments of L-functions to show that at least $N/(\log N)^2$ are nonzero at $s = 1$, which along with Kolyvagin's results may be used to give a corresponding improvement in the answer to the first question.

One can also use Riemann's explicit formula for the L_f's (see Mestre [11]) to control the average order at $s = 1$, although this approach requires (at least at present) an appeal to the Riemann hypothesis to obtain the desired bounds. In recent papers, Brumer [1] and Murty [12] have used these techniques to show that (assuming the

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