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VOLUME GROWTH, GREEN'S FUNCTIONS, AND PARABOLICITY OF ENDS

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§1. Introduction. Let M be a complete, noncompact Riemannian n-manifold ($n \geq 2$) without boundary. We fix a point $o \in M$ once and for all, and we write $|x| = d(x,o)$ for the distance between points x and o and write $V(r) = |B(o,r)|$ for the volume of the geodesic ball of radius r centered at o. The purpose of this paper is to find conditions on M in order to characterize the existence of (positive) Green's functions for the p-Laplace equation on M in terms of the growth of $V(r)$. In this respect, the paper is closely related to [LT1] and [LT3], where similar questions were studied for the usual Laplace equation. However, our approach is more general in two aspects. First, the equations whose solutions are studied form a wide class, including the Laplace equation as a special case. On the other hand, our assumptions on manifolds are milder than those in [LT1] and [LT3]; thus, even in the case of harmonic functions, we get new results. It is worth pointing out that we do not make any curvature assumptions. We require instead that certain analytic and geometric inequalities hold on M.

If $C \subset M$ is a compact set, then an unbounded component of $\complement C = M \setminus C$ is called an *end* with respect to C. Throughout the paper, we assume that M has *finitely many ends*, which means that the number of ends with respect to any compact set has a uniform upper bound. Thus there exist an integer m and a compact set C such that M has exactly m ends with respect to any other compact set C' that contains C. We fix such a compact set C and positive R_0 such that $C \subset \bar{B}(o, R_0)$. From now on, E will be an unbounded component of $\complement C$.

Given $x_0 \in M, R > 0$, and $p \geq 1$, we assume that a *volume-doubling property* and a *weak* $(1, p)$*-Poincaré inequality* hold with constants $C_1 = C_1(x_0, R)$ and $C_2 = C_2(x_0, R, p)$ inside the ball $B(x_0, R)$. By these we mean that for every ball $B = B(x,r)$, with $2B = B(x, 2r) \subset B(x_0, R)$,

$$|2B| \leq C_1 |B| \tag{1.1}$$

and that

$$\fint_B |u - u_B| \leq C_2 r \left(\fint_{2B} |\nabla u|^p \right)^{1/p}, \quad \text{with } u_B = \fint_B u = \frac{1}{|B|} \int_B u, \tag{1.2}$$

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