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ON THE GAMMA FACTOR OF THE TRIPLE L-FUNCTION, I

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Introduction. The study of the triple L-function began with Garrett [2]. From a representation-theoretic point of view, Piatetski-Shapiro and Rallis [9] and the author [6] defined the archimedean or nonarchimedean L-factor as the greatest common divisor (GCD) of local integrals. The purpose of this paper is to prove that the archimedean GCD L-factor agrees with the expected L-factor up to invertible functions if the local representation comes from a cuspidal automorphic representation. This theorem follows from the local functional equation by formal argument except when the representation is unramified. To treat the unramified case, we carry out the explicit calculation of the unramified local integral. Stimulated by Stade [12], we make use of a hypergeometric function ${}_3F_2$ and its two-term relations and three-term relation. We also need a theorem on some infinite integral involving Bessel functions, proved by Bailey [1] in 1936. The key lemma is the following (see Lemma 2.4).

Lemma. *For* $\mathrm{Re}(s) > |\mathrm{Re}(\sigma_1)| + |\mathrm{Re}(\sigma_2)| + |\mathrm{Re}(\sigma_3)|$,

$$\int_0^\infty \int_0^\infty \int_0^\infty K_{2s}(x+y+z) K_{2\sigma_1}(x) K_{2\sigma_2}(y) K_{2\sigma_3}(z) \cdot x^{2s} y^{2s} z^{2s} (x+y+z)^{-2s} d^\times x\, d^\times y\, d^\times z$$

is equal to

$$2^{4s-5}\Gamma(4s)^{-1} \prod_{e_1,e_2,e_3\in\{\pm 1\}} \Gamma(s+e_1\sigma_1+e_2\sigma_2+e_3\sigma_3).$$

In §1, we review the theory of triple L-functions. In §2, the calculation of the unramified local integral is reduced to Lemma 2.4. In §3, we prove Lemma 2.4.

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§1. Review of theory of triple L-functions. We review the theory of triple L-functions (see [6], [9]). Let k be an algebraic number field and $\mathbb{A}$ be the adèle ring of k. We fix a nontrivial additive character ψ of $k\backslash\mathbb{A}$. Let G be an algebraic group defined by

$$G = \left\{ g = (g_1, g_2, g_3) \in (\mathrm{GL}_2)^3 \,\middle|\, \det g_1 = \det g_2 = \det g_3 \right\}.$$

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