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ON THE EXISTENCE OF MULTIPEAKED SOLUTIONS TO A SEMILINEAR NEUMANN PROBLEM

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1. Introduction and statement of results. The aim of this paper is to establish the existence of single- and multipeaked positive solutions to the problem

$$(\mathbf{P})_\varepsilon \qquad \begin{cases} -\varepsilon^2 \Delta u + u = u^{p-1}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where Δ is the Laplace operator, Ω is a smooth bounded domain in $\mathbf{R}^N$ $(N \geq 2)$, ν is the unit outward normal vector to $\partial\Omega$, $2 < p < 2N/(N-2)$ for $N \geq 3$, $2 < p < +\infty$ for $N = 2$, and $\varepsilon > 0$ is a constant.

Problem $(\mathbf{P})_\varepsilon$ may be viewed as prototype of pattern formation in biology. Indeed, the steady-state problem for a chemotactic aggregation model with logarithmic sensitivity is reduced by Keller and Segel to $(\mathbf{P})_\varepsilon$ (see [20]). Moreover, in the study of activator-inhibitor systems modeling biological pattern formation, proposed by Gierer and Meinhardt in [17], $(\mathbf{P})_\varepsilon$ plays an important role when the diffusion rate of the inhibitor is sufficiently large. See, for example, [24], [33], and the references therein for more details.

One of the motivations of this paper is the "point-condensation phenomena" of solutions to $(\mathbf{P})_\varepsilon$ expected from numerical simulations (to the Keller-Segel model as well as to the Gierer-Meinhardt model). That is, the solutions to $(\mathbf{P})_\varepsilon$ seem to tend to zero as $\varepsilon \longrightarrow 0$ except at a finite number of points. Here we show that these points are determined as local maximun and minimum points of the mean curvature of the boundary $\partial\Omega$.

The results are obtained by variational methods. Define an "energy"

$$J_\varepsilon : H^1(\Omega) \longrightarrow \mathbf{R}$$

associated with $(\mathbf{P})_\varepsilon$ by

$$J_\varepsilon(v) = \int_\Omega \left[\frac{1}{2}\left(\varepsilon^2 |\nabla v|^2 + |v|^2 \right) - \frac{1}{p} v_+^p \right] dx, \tag{1.1}$$

where $v_+ = \max(v, 0)$. A solution u of $(\mathbf{P})_\varepsilon$ is called a least-energy solution if it has the smallest energy $J_\varepsilon(u)$ among all solutions (not identically zero) to $(\mathbf{P})_\varepsilon$. Using

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