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DESCRIPTION OF THE n-ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS AND HAMILTONIAN INTEGRABLE SYSTEMS OF HYDRODYNAMIC TYPE, I: INTEGRATION OF THE LAMÉ EQUATIONS

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1. Introduction. The problem of describing n-orthogonal curvilinear coordinate systems can be formulated as follows: Find in R^n all the coordinate systems

$$u^i = u^i(x^1, \ldots, x^n), \tag{1.1}$$

$$\det\left\|\frac{\partial u^i}{\partial x^j}\right\| \neq 0, \tag{1.2}$$

satisfying the condition of orthogonality

$$\sum_{k=1}^{n} \frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^k} = 0, \qquad i \neq j. \tag{1.3}$$

The problem can be formulated either locally (in the same domain Ω) or globally (in the whole R^n). In the latter case, one can admit that condition (1.2) can be violated on some manifold of dimension $m < n$, and the system of intersecting hypersurfaces may have a nontrivial topology. Coordinates $u^i(x)$ are defined up to an obvious transformation

$$u^i = f^i(\tilde{u}^i). \tag{1.4}$$

For $n = 2$, the problem can be solved very easily. Let us choose a function (u^1, for instance) in an arbitrary way and consider a system of its level lines on the plane x^1, x^2. Then one can construct the vector field of normals to the level lines. Integral curves of this vector field are the level lines for u^2, which can be reconstructed uniquely up to transformation (1.4).

For $n \geqslant 3$, the problem is much more difficult. The first nontrivial case $n = 3$ is known in differential geometry as the problem of triply orthogonal systems of surfaces. It was formulated in 1810 when Dupin and Binet found a family of

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