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THE SPACE OF RATIONAL MAPS ON $\mathbb{P}^1$

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CONTENTS

The set of morphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d is parametrized by an affine open subset Rat_d of $\mathbb{P}^{2d+1}$. In this paper, we consider the action of SL_2 on Rat_d induced by the *conjugation action* of SL_2 on rational maps; that is, $f \in \mathrm{SL}_2$ acts on ϕ via $\phi^f = f^{-1} \circ \phi \circ f$. The quotient space $M_d = \mathrm{Rat}_d/\mathrm{SL}_2$ arises very naturally in the study of discrete dynamical systems on $\mathbb{P}^1$. We prove that M_d exists as an affine integral scheme over $\mathbb{Z}$, that M_2 is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$, and that the natural completion of M_2 obtained using geometric invariant theory is isomorphic to $\mathbb{P}^2_{\mathbb{Z}}$. These results, which generalize results of Milnor over $\mathbb{C}$, should be useful for studying the arithmetic properties of dynamical systems.

§1. Notation and summary of results. A rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d over a field K is given by a pair of homogeneous polynomials

$$\phi = [F_a, F_b] = [a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d, b_0X^d + b_1X^{d-1}Y + \cdots + b_dY^d]$$

of degree d such that F_a and F_b have no common roots (in $\mathbb{P}^1(\bar{K})$). This last condition is equivalent to the condition that

$$\mathrm{Res}(F_a, F_b) \neq 0,$$

where the resultant $\mathrm{Res}(F_a, F_b)$ is a certain bihomogeneous polynomial in the coefficient $a_0, a_1, \ldots, a_d, b_0, \ldots, b_d$. We also frequently write such maps ϕ in non-homogeneous form as

$$\phi(z) = \frac{a_0z^d + a_1z^{d-1} + \cdots + a_{d-1}z + a_d}{b_0z^d + b_1z^{d-1} + \cdots + b_{d-1}z + b_d}.$$

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