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EXISTENCE OF A COMPLEX LINE IN TAME ALMOST COMPLEX TORI

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0. Introduction

Motivation. A complex line in a complex space V is a nonconstant holomorphic map from $\mathbb{C}$ to V. If V is hyperbolic (i.e., if the Kobayashi pseudodistance between different points of V is positive), then V does not admit a complex line. For compact complex manifolds M, R. Brody [Br] proved the converse: If M is nonhyperbolic, then M admits a complex line. Brody's result generalizes to compact complex spaces; see [L, Chapter III]. Building on M. Gromov's existence results [G] for pseudoholomorphic curves and on Brody's idea, we pursue this question into the realm of almost complex manifolds. Following Gromov we say that a symplectic form ω on a manifold M tames an almost complex structure J on M if $\omega(v, Jv) > 0$ for all nonzero $v \in TM$. If M is a torus T^{2n}, we call a symplectic structure standard if—up to a diffeomorphism of T^{2n}—its pullback to the universal cover $\mathbb{R}^{2n}$ is a constant form. Our main result is the following theorem.

THEOREM 0.1. *Let J be an almost complex structure on T^{2n} that is tamed by a standard symplectic structure. Then there exists a nonconstant J-holomorphic $f : \mathbb{C} \to T^{2n}$ with uniformly bounded differential.*

The motivation for this result is a question posed by J. Moser in [Mo]. In this paper Moser proves a KAM-type perturbation result, stating that many of the holomorphic foliations (by affine complex lines) of a standard complex torus "persist" if the complex structure is slightly perturbed to an almost complex one. He asks if (in real dimension 4) there is a—possibly weaker—version of his result that holds for large perturbations of the complex structure. Clearly Theorem 0.1 is a first step in this direction. Further steps are planned.

To understand Moser's question, one has to see the analogy of this problem with the problem of minimal geodesics in Riemannian tori that is explained in [Mo, 1.g]. For further information see [Ba] and [Ma]. Below we give an outline of the proof of Theorem 0.1. In a structural sense this proof resembles the (simple) proof that every compact Riemannian manifold (M, g) with infinite fundamental group has a minimal geodesic (or "geodesic line"); that is, a complete nonconstant geodesic whose lift to the universal cover is the shortest connection between any two of its points.

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