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A GEOMETRIC PROOF OF THE CIRCULAR MAXIMAL THEOREM

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1. Introduction. A well-known result of Bourgain [1] asserts that the circular maximal function

$$\mathcal{M}f(x) = \sup_{1<t<2} \int_{S^1} |f(x - ty)| \, d\sigma(y) \tag{1.1}$$

is bounded on $L^p(\mathbb{R}^2)$ for $p > 2$. Simple examples show that this fails for $p = 2$. In this paper, we derive Bourgain's result by geometric and combinatorial methods. In particular, we do not use the Fourier transform in any way. Our proof is based on a combinatorial argument from [7], which in turn uses Marstrand's three circle lemma [8], and a lemma involving two circles that seems to originate in [11]. The three circle lemma was used in [8] to prove the following result, which is a simple consequence of Bourgain's theorem.

Suppose a planar set E has the following property: for every point in the plane, E contains some circle with that point as center. Then E has positive measure.

Thus we show here that Marstrand's lemma, in combination with other ideas, does indeed allow one to establish the stronger maximal function estimate. Furthermore, we demonstrate in Section 4 how to obtain the entire known range of $L^p \to L^q$ estimates for the circular maximal function (which is optimal possibly up to endpoints) (see [10] and [12]), by using the methods from Sections 2 and 3. One—perhaps significant—distinction from the techniques developed in [1], [9], [12], and [13], which involve the Fourier transform, is the fact that the methods presented here do not seem to yield estimates for the global maximal function

$$\overline{\mathcal{M}}f(x) = \sup_{0<t<\infty} \int_{S^1} |f(x - ty)| \, d\sigma(y).$$

It is well known that for a general class of maximal functions, L^p-boundedness is equivalent to an L^q-covering property for the associated geometric family (see [5]). Moreover, Córdoba [4] posed the problem of finding a geometric

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