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AN INDEX FOR COUNTING FIXED POINTS OF AUTOMORPHISMS OF FREE GROUPS

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Introduction. Let α be an automorphism of $F = F_n$, the free group of rank n. The Scott conjecture, proved by Bestvina-Handel [BH], states that the fixed subgroup $\operatorname{Fix}\alpha = \{g \in F \mid \alpha(g) = g\}$ has rank at most n.

Using **R**-trees, we shall improve this result by showing the following theorem.

THEOREM 1. *If α is any automorphism of F_n, then* $\operatorname{rk}\operatorname{Fix}\alpha + a(\alpha)/2 \leqslant n$.

Here $a(\alpha)$ is the number of *equivalence classes of attracting fixed points* for the action of α on the boundary of F (defined below). This positively answers a conjecture of Cooper [Co, p. 455].

If $\operatorname{Fix}\alpha$ is trivial, our result specializes to the following corollary.

COROLLARY 2. *An automorphism α of F_n with* $\operatorname{Fix}\alpha = \{1\}$ *fixes at most* $4n$ *ends of F_n.*

To define $a(\alpha)$, in general, we consider the boundary δF of F (see Section 1), the Cantor set of ends of F if $n \geqslant 2$. If we choose a free basis $g_1, \ldots, g_n$, it may be viewed as the set of all infinite reduced words $X = x_1 \cdots x_i \cdots$ in the letters $g_j^{\pm 1}$. The action of α on F extends to a continuous action of α on δF. The boundary of the subgroup $\operatorname{Fix}\alpha$ naturally embeds in δF, and α acts on $\delta(\operatorname{Fix}\alpha)$ as the identity.

We consider fixed points of α in δF. It turns out (Proposition 1.1) that such a fixed point X either belongs to $\delta(\operatorname{Fix}\alpha)$, or is attracting, or is repelling (i.e., attracting for α^{-1}). Here *attracting* may be understood in the topological sense ($\lim_{p\to+\infty} \alpha^p(X') = X$ for X' close to X in $F \cup \delta F$), or in the algebraic sense of [CL1, (1.4)]. As in [CL1], we say that two fixed points $X_1, X_2 \in \delta F$ are *equivalent* if there exists $g \in \operatorname{Fix}\alpha$ such that $X_2 = gX_1$. Note that any point equivalent to an attracting fixed point of α is itself an attracting fixed point of α.

We let $\mathscr{A}(\alpha)$ be the set of equivalence classes of attracting fixed points of α, and we denote $a(\alpha)$ the cardinality of $\mathscr{A}(\alpha)$. The finiteness of $a(\alpha)$ follows from [Co] (or [CL1]).

Theorem 1 may be illustrated by the following example from [CL1]. Let $\alpha : F_2 \to F_2$ be given by $\alpha(a) = aba$, $\alpha(b) = ba$. The fixed subgroup has rank 1 and it is generated by $aba^{-1}b^{-1}$. One obtains two inequivalent fixed words $X_1 = ababaaba \cdots$ and $X_2 = a^{-1}b^{-1}a^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1} \cdots$ by taking the limit as p goes to $+\infty$ of $\alpha^p(a)$ and $\alpha^p(a^{-1})$, respectively. Note that $X_3 = baabaaba \cdots = \lim_{p\to\infty} \alpha^p(b)$ is equivalent to X_1. The automorphism α is induced

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