Vol. 93, No. 2 DUKE MATHEMATICAL JOURNAL © 1998

EIGENFUNCTION DECAY ESTIMATES IN THE QUANTUM INTEGRABLE CASE

JOHN A. TOTH

1. Introduction. The motivation for embarking on this work stems, in large part, from the work of Martinez [M1]–[M4], Wilkinson [Wi], and from our previous work [T1]–[T3].

Let X be a compact C^∞ manifold. There have been many papers written on the construction of approximate eigenfunctions for a given selfadjoint elliptic partial differential operator $P: C^\infty(X) \to C^\infty(X)$ (see [C], [G], [GSt], [GW], [R], [Sa], and the references therein). These constructions are usually microlocal variants on the classical WKB ansatz. That is, one constructs an asymptotic solution $u_\lambda(x)$ to the equation

$$Pu_\lambda = \lambda u_\lambda$$

as a superposition of oscillatory integrals of the form

$$\int e^{i\lambda\phi(x,\theta)} a(x,\theta;\lambda)\, d\theta,$$

where $a(x,\theta;\lambda) \sim a_0(x,\theta) + \lambda^{-1}a_1(x,\theta) + \cdots$. Here, the $a_j(x,\theta)$ solve a sequence of transport equations, and $\phi(x,\theta)$ solves an eikonal equation. Such techniques have proved quite successful for studying eigenvalue asymptotics. However, as is well known, these quasi modes are not necessarily close to true eigenfunctions in any reasonable sense. To prove results about eigenfunctions, one is usually required to establish the a priori existence of spectral gaps; this problem is, in most instances, quite intractable. It thus seems desirable to study eigenfunctions directly. This is what we shall do in the case of a real-analytic quantum integrable system. Although, at first glance, such systems seem unduly restrictive, they include many of the known finite-dimensional, classical integrable systems.

Henceforth, we assume that X is a compact, real-analytic manifold, equipped with a real-analytic, Riemannian metric. Let $P_1, \ldots, P_n$ be n pairwise-commuting, selfadjoint, $\hbar$-analytic pseudodifferential operators on X, which are, in addition, jointly elliptic (i.e., $\sum_{k=1}^n P_k^2$ is elliptic in the standard sense). We denote by

Received 2 May 1996. Revision received 27 February 1997.

The author was supported in part by Natural Sciences and Engineering Research Council grant number OGP0170280 and Fonds pour la Formation de Chercheurs et l'Aide à la Recherche grant number NC-1520.

1991 *Mathematics Subject Classification.* Primary 58FO7, 58G25; Secondary 33E10.