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THE STABLE 4-DIMENSIONAL GEOMETRY OF THE REAL GRASSMANN MANIFOLDS

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0. Introduction. In their fundamental paper *Calibrated geometries* in 1982 [HL], R. Harvey and B. Lawson, discussing future global applications, posed the problem of determining the subvarieties of the real Grassmann manifolds G_kR^n of oriented k-planes through the origin in R^n which can be shown to be volume-minimizing in their homology classes by using as calibrations the invariant forms representing the universal Pontryagin classes.

We carry out this method here for the first Pontryagin form, using it as a calibration to determine volume-minimizing 4-dimensional cycles in all real Grassmann manifolds. We also learn that their 4-dimensional geometry stabilizes immediately after the Grassmannian G_4R^8.

Main Theorem. (a) *The comass of the first Pontryagin form on the Grassmann manifold G_mR^{m+n} stabilizes at the value $3/2$ for $m, n \geqslant 4$.*

(b) *When $m, n \geqslant 4$, this maximum value of $3/2$ is achieved on the $O(m) \times O(n)$ orbit of a single oriented 4-plane in the tangent space to $G_4R^8 \subset G_mR^{m+n}$ at one point, and nowhere else.*

(c) *The round 4-spheres in G_4R^8, which represent base spaces of Hopf fibrations of S^7 by great 3-spheres, and which also minimize volume in their homology classes there, remain volume-minimizing in all higher Grassmannians.*

As a result, we see that the 4-dimensional geometry of the real Grassmann manifolds, as expressed through their volume-minimizing 4-dimensional subvarieties, varies just like their 4-dimensional topology; it rises to a crescendo at G_4R^8, then immediately diminishes and stabilizes forever.

As usual in this subject, the technically most demanding part of the proof involves the computation of the comass of the first Pontryagin form. This is carried out by introducing a new *quaternionic symmetrization* scheme, in which the computation on G_mR^{m+n} is "reduced" to a corresponding computation, in a more symmetrical setting, on G_mR^{m+4n}.

By using this symmetrization technique, I was able to reduce the original problem of a quartic form to a problem of a quadratic form. I then systematically calculated the comasses of the first Pontryagin forms of all G_mR^{m+n}. The results gave the correct comasses of the first Pontryagin forms for small m and n in [GMM]. For $m > 4$ and $n > 4$, the results are new. Whether the infinitesimal uniqueness result of part (b) of the theorem can be promoted to a global uniqueness result (which asserts that for Grassmann manifolds beyond G_4R^8, the only