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HIGHEST WEIGHT MODULES OVER THE $W_{1+\infty}$ ALGEBRA AND THE BISPECTRAL PROBLEM

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0. Introduction. This paper is part of a series of papers [5]–[8] in which we study different aspects of the *bispectral problem*. As originally formulated by J. J. Duistermaat and F. A. Grünbaum [14], this problem asks for which ordinary differential operators $L(x, \partial_x)$ there exists a nonzero family of eigenfunctions $\psi(x,z)$, which are also eigenfunctions of another differential operator $\Lambda(z, \partial_z)$ in the spectral parameter z. That is, for which L, Λ, and ψ the following identities hold:

$$L(x, \partial_x)\psi(x,z) = f(z)\psi(x,z), \tag{0.1}$$

$$\Lambda(z, \partial_z)\psi(x,z) = \theta(x)\psi(x,z), \tag{0.2}$$

with some functions $f(z)$ and $\theta(x)$. Both L and Λ are called *bispectral operators.*

Initially, the bispectral problem was connected with certain studies in computer tomography (cf. [14]). Later, it turned out to be linked to several actively developing areas of mathematical physics and, in particular, to soliton mathematics. With the present paper, we also establish such connections—this time with the Lie algebra $W_{1+\infty}$ and its subalgebras.

In order to place our work properly among the other research, we first recall the main results of the pioneering paper [14] where Duistermaat and Grünbaum classified all second order bispectral operators L. The complete list is as follows. If we present L as a Schrödinger operator

$$L = \left(\frac{d}{dx}\right)^2 + u(x),$$

the bispectral potentials $u(x)$, apart from the obvious Airy ($u(x) = ax$) and Bessel ($u(x) = cx^{-2}$) ones, are organized into two families of potentials $u(x)$, which can be obtained by finitely many "rational Darboux transformations"

(1) from $u(x) = 0$,
(2) from $u(x) = -(1/4)x^{-2}$.

An important difference between the two families is that while in the first case the dimension of the space of eigenfunctions $\psi(x,z)$ of (0.1) and (0.2) is 1, in the

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