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THE GAUSS-MANIN CONNECTION OF THE INTEGRAL OF THE DEFORMED DIFFERENCE PRODUCT

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To the Memory of Michitake Kita

1. Introduction. Let $I = \{(i,j) | 1 \leqslant i \leqslant m, \ \ m+1 \leqslant j \leqslant N \ \text{ or } \ m+1 \leqslant i < j \leqslant N\}$. We consider the integral

$$\int \prod_{i=m+1}^{N} t_i^{\alpha_i} \prod_{(i,j)\in I} (x_{ji}t_i - x_{ij}t_j)^{\beta_{ij}} \, dt_{m+1} \wedge \cdots \wedge dt_N \tag{1.1}$$

over a suitable cycle, where we understand that $t_1 = \cdots = t_m = 1$. As a function of x_{ij}, this integral satisfies a linear integrable Pfaffian system called the Gauss-Manin connection, and the purpose of this paper is to give an explicit formula of it. We shall denote by Φ the *deformed difference product* $\prod_{i=m+1}^{N} t_i^{\alpha_i} \prod_{(i,j)\in I}(x_{ji}t_i - x_{ij}t_j)^{\beta_{ij}}$. For the sake of simplicity we understand that $\beta_{ij} = \beta_{ji}$.

This type of integral was first studied by Baldassarri. In [B], he was led to this type of integral by seeking to apply Dwork's theory of generalized hypergeometric functions (see [D]) to p-adic interpolation of Evans character sums. Our interest stems from the observation of Sturmfels (cited in [B]) that when $m = 1$, the rank of the A-hypergeometric system of the integral is N^{N-2}. This is the number of (labeled) rooted trees with $N-1$ edges, and this same number appears as the rank of the q-twisted cohomology group associated with the q-difference product (see [A3]), in which the basis of the cohomology group is identified with rooted trees. Hence it seems natural to guess that the twisted de Rham rational cohomology group $H^{N-m}(M, \nabla_\omega)$ (see Section 2) associated with our deformed difference product has a basis, each element of which is identified with a rooted tree in the case $m = 1$, and when $m \geqslant 2$, it is identified with a *forest* (i.e., a disjoint union of rooted trees). We show this is indeed the case (see Theorem 2.7), and with this basis we can calculate the Gauss-Manin connection explicitly (see Theorem 3.1). In Section 4, we deal with a degenerate case treated by Aomoto [A2] in a parallel way with the deformed case. It seems to be more transparent both in understanding and in calculation to proceed from the (easier) deformed case to a degenerate case than to treat the latter by itself. We note that our proof of Theorem 4.3 gives a correct proof for the corollary of Lemma 1.2 in [A2], which describes a basis of $H^{N-m}(M, \nabla_\omega)$ (see Remark 1 after Proposition 4.4).

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