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PREHOMOGENEOUS VECTOR SPACES AND ERGODIC THEORY, I

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Introduction. This is the first in a series of papers in which we consider problems analogous to the Oppenheim conjecture from the viewpoint of prehomogeneous vector spaces.

Throughout this paper, k is a field of characteristic zero. The following theorem, known as the Oppenheim conjecture, was proved by Margulis [18].

THEOREM 0.1 (Margulis). *Let Q be a real nondegenerate indefinite quadratic form in $n \geqslant 3$ variables. Suppose that the corresponding point in $\mathbb{P}(\mathrm{Sym}^2(\mathbb{R}^n)^*)$ is irrational. Then the set of values of Q at primitive integer points is dense in $\mathbb{R}$.*

The above theorem for $n \geqslant 5$ was conjectured by Oppenheim in [20]. Margulis originally proved that values of Q at integer points can be arbitrarily small (nontrivially, of course), which implies that the set of values of Q at integer points is dense in $\mathbb{R}$ due to the result of Lewis [17]. The above improved version (the primitive part) is due to Dani-Margulis [7]. A further improvement of this result was obtained by Borel-Prasad [3]. Some partial results were known prior to the work of Margulis.

Let $f(x)$ be a degree d form in real n variables $x = (x_1, \dots, x_n)$. In the following, we always assume that f is not a multiple of an integral form. Consider the following questions.

(1) For any $\varepsilon > 0$, does there exist $x \in \mathbb{Z}^n \backslash \{0\}$ such that $|f(x)| < \varepsilon$?
(2) For any $\varepsilon > 0$, does there exist $x \in \mathbb{Z}^n \backslash \{0\}$ such that $0 < |f(x)| < \varepsilon$?
(3) Is the set $\{f(x) | x \in \mathbb{Z}^n\}$ dense in $\mathbb{R}$?

For nondegenerate quadratic forms, one typically gets stronger results such as (2) or (3). For forms of higher degree, there is no notion of "nondegenerate" forms. However, for (1), this is not necessary. For example, if the form does not

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