Vol. 89, No. 3 DUKE MATHEMATICAL JOURNAL © 1997

ON ISOMETRIC AND MINIMAL ISOMETRIC EMBEDDINGS

THOMAS IVEY AND J. M. LANDSBERG

Introduction

Definition. Let $(M^n, \tilde{g})$ be a Riemannian manifold. We will say $\tilde{g}$ is a *quasi-κ-curved metric* if there exists a smooth positive definite quadratic form Q on M such that for all $x \in M$

$$R_x = -\gamma(Q_x, Q_x) + (\kappa + 1)\gamma(\tilde{g}_x, \tilde{g}_x), \tag{1}$$

where $\gamma\colon S^2T^* \to S^2(\Lambda^2 T^*)$ denotes the algebraic Gauss mapping and R_x the Riemann curvature tensor. (See §1 for more details.)

Quasi-κ-curved metrics are a generalization of *quasi-hyperbolic metrics*, defined in [BBG], which correspond to $\kappa = -1$. We will also refer to the case $\kappa = 0$ as *quasi-flat metrics*. We will assume that $n \geqslant 3$. When $n = 3$, the quasi-κ-curved condition is an open condition on the metric, and thus in this case, the class of metrics we study is quite general. The condition is stronger in higher dimensions.

In this paper we study local isometric embeddings and minimal isometric embeddings of quasi-κ-curved manifolds. Before giving our results, it will be useful to review some of what is known.

Local isometric embeddings. Given a Riemannian manifold $(M^n, \tilde{g})$, one may ask if it admits a local isometric embedding into a Euclidean space $\mathbb{R}^{n+r}$ or, more generally, a space form $X(\varepsilon)^{n+r}$ of constant sectional curvature ε. If M is more positively curved than X, one expects to have local isometric embeddings (e.g., the embedding of S^n into $\mathbb{R}^{n+1}$). We will be concerned with the case where M is less positively curved than X.

The critical codimension for the isometric embedding problem is $r = \binom{n}{2}$. The Cartan-Janet theorem states that local isometric embeddings of analytic Riemannian n-folds into $\mathbb{R}^{n+\binom{n}{2}}$ exist and depend locally on a choice of n arbitrary functions of $n-1$ variables. (These "dimension counts" come from the Cartan-Kähler theorem [Car2].) In this paper, we will be interested in the overdetermined case $r < \binom{n}{2}$.

In the most overdetermined case ($r = 1$), Thomas [T] observed that the Codazzi equations of a hypersurface with nondegenerate second fundamental form are consequences of the Gauss equations when $\dim(M) \geqslant 4$. Thus, codi-

Received 10 August 1995. Revision received 31 July 1996.
The second author is partially supported by National Science Foundation grant DMS-9303704.