Vol. 89, No. 1 DUKE MATHEMATICAL JOURNAL © 1997

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0. Introduction. The Uniformization Program of William Thurston prescribes, for a large class of compact, closed three-manifolds, the existence of a hyperbolic metric, i.e., a metric of constant negative curvature -1. One approach to finding such a metric is a two-step variational problem, which we now describe. Consider a compact Riemannian manifold (M^n, g), and for any metric $\tilde{g}$ in the conformal class $C = C(g)$, define

$$Y(\tilde{g}) = \frac{\int_M R_{\tilde{g}} d\,\mathrm{Vol}_{\tilde{g}}}{[\mathrm{Vol}(\tilde{g})]^{n-2/n}}, \tag{1}$$

where $R_{\tilde{g}}$ is the scalar curvature. Then one easily shows that the critical points $\tilde{g}$ of $Y(\tilde{g})$ in C are metrics of constant scalar curvature [18]. The minimum $\inf Y(\tilde{g})$, $\tilde{g} \in C$ is called the Yamabe invariant of (M, g) and is denoted $\lambda(g)$. The solution to the Yamabe problem, obtained in the last thirty years, due to the efforts of Yamabe [18], Trudinger [16], Aubin [1], and Schoen [12], [13], establishes the existence of the minimizing metric $\tilde{g}$ in C. For M three-dimensional, one knows, under certain topological restrictions, that the constant scalar curvature of a metric on M may not be positive. In fact, if in the prime decomposition of M there is a factor that is not covered by a homotopy sphere or a torus and is not diffeomorphic to $S^1 \times S^2$, (say, M is hyperbolizable), then M does not carry a metric of nonnegative scalar curvature, as shown by Schoen-Yau [14] and Gromov-Lawson [4]. So for such manifolds, $\lambda(g)$ is always negative. The importance of the functional $C \to \lambda(g)$ stems from the hope to find a conformal class C, maximizing this functional. Then an easy computation shows that the corresponding constant scalar curvature metric $\tilde{g}$ in C is actually Einstein [12], hence hyperbolic, since $\dim M = 3$. For such manifolds, it is more convenient to introduce a modified function $\mathrm{Vol}\colon C \mapsto \mathrm{Vol}(C)$, as follows: Take the appropriately scaled metric $\tilde{g}$ in C of constant scalar curvature -1, and denote $\mathrm{Vol}(C) = \mathrm{Vol}(\tilde{g})$. We will call the set $\{\mathrm{Vol}(C)\}$ the Yamabe spectrum of M.

The expectation for the global minimum of Vol at the hyperbolic metric is justified by the following known facts. First, if g_0 is a (necessarily unique up to a diffeomorphism) hyperbolic metric on M, then $(D^2\,\mathrm{Vol})_{g_0} > 0$; see [5, Theorem 8.2]. Second, if $\dim M = 4$ and g_0 is hyperbolic, then the Euler characteristic and signature computations of Johnson and Millson [5, Theorem 8.3] show that Vol attains its global minimum at g_0.

Received 1 August 1995.