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MOUFANG TREES AND GENERALIZED OCTAGONS

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1. Introduction. Let Γ be an undirected graph, let $V(\Gamma)$ denote the vertex set of Γ, and let G be a subgroup of $\mathrm{aut}(\Gamma)$. For $x \in V(\Gamma)$, we will denote by Γ_x the set of vertices adjacent to x in Γ and by $G_x^{[i]}$ for any $i \geqslant 1$ the pointwise stabilizer in G_x of the set of vertices of Γ at distance at most i from x. Thus $G_x^{[1]}$ is the kernel of the action of G_x on Γ_x. An n-path of Γ for any $n \geqslant 0$ is an $(n+1)$-tuple $(x_0, x_1, \ldots, x_n)$ of vertices such that $x_i \in \Gamma_{x_{i-1}}$ for $1 \leqslant i \leqslant n$ and $x_i \neq x_{i-2}$ for $2 \leqslant i \leqslant n$. Let

$$G_{x,y,\ldots,z}^{[i]} = G_x^{[i]} \cap G_y^{[i]} \cap \cdots \cap G_z^{[i]}$$

for any subset $\{x, y, \ldots, z\}$ of $V(\Gamma)$ and any $i \geqslant 1$. The graph Γ will be called thick if $|\Gamma_x| \geqslant 3$ for every $x \in V(\Gamma)$. An apartment of Γ is a connected subgraph Ξ such that $|\Xi_x| = 2$ for every $x \in V(\Xi)$. When there is no danger of confusion, we will often use integers to denote vertices of Γ.

A generalized n-gon (for $n \geqslant 2$) is a bipartite graph of diameter n and girth $2n$. A generalized n-gon Γ for $n \geqslant 3$ is called Moufang if $G_{1,\ldots,n-1}^{[1]}$ acts transitively on $\Gamma_n \backslash \{n-1\}$ for every $(n-1)$-path $(1, \ldots, n)$ of Γ for some $G \leqslant \mathrm{aut}(\Gamma)$. In [7], Tits showed that thick Moufang n-gons exist only for $n = 3$, 4, 6, and 8. If Γ is a generalized n-gon and $G \leqslant \mathrm{aut}(\Gamma)$, then $G_{0,1}^{[1]} \cap G_{0,\ldots,n} = 1$ for every n-path $(0, \ldots, n)$ of Γ. (This is a special case of [6, (4.1.1)]; see Theorem 2 of [10].) Thus, the following (Theorem 1 of [11]) is a generalization of Tits's result.

1.1. Theorem. *Let Γ be a thick connected graph, let $G \leqslant \mathrm{aut}(\Gamma)$, and let $n \geqslant 3$. Suppose that for each n-path $(0, 1, \ldots, n)$ of Γ,*
(i) *$G_{1,\ldots,n-1}^{[1]}$ acts transitively on $\Gamma_n \backslash \{n-1\}$, and*
(ii) *$G_{0,1}^{[1]} \cap G_{0,\ldots,n} = 1$.*
Then $n = 3$, 4, 6, or 8.

We call a graph Γ (G, n)-Moufang if it is thick and connected and if Γ, G, and n fulfill conditions (i) and (ii) of Theorem 1.1. In this paper, we will be mainly concerned with the case that Γ is a tree.

In [1, (3.6)], the following beautiful connection between trees and generalized polygons was established; see also §6 of [12].

1.2. Theorem. *Let $n \geqslant 3$. Suppose Γ is a tree and $\mathcal{A}$ a family of apartments of Γ such that*

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