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STABILITY OF THE BLOW-UP PROFILE FOR EQUATIONS OF THE TYPE $u_t = \Delta u + |u|^{p-1}u$

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1. Introduction. In this paper, we are concerned with the following nonlinear equation:

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u \\ u(.,0) &= u_0 \in H, \end{aligned} \tag{1}$$

where $u(t)\colon x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, Δ stands for the Laplacian in $\mathbb{R}^N$. We note $H = W^{1,p+1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We assume in addition the exponent p subcritical: if $N \geqslant 3$, then $1 < p < (N+2)/(N-2)$; otherwise, $1 < p < +\infty$. Other types of equations will be also considered.

The local Cauchy problem for equation (2) can be solved in H. Moreover, one can show that either the solution $u(t)$ exists on $[0,+\infty)$, or on $[0,T)$ with $T < +\infty$. In this former case, u blows up in finite time in the sense that

$$\|u(t)\|_H \to +\infty \quad \text{when } t \to T.$$

(Actually, we have both $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to +\infty$ and $\|u(t)\|_{W^{1,p+1}(\mathbb{R}^N)} \to +\infty$ when $t \to T$.)

Here we are interested in blow-up phenomena. (For such a case, see, for example, Ball [1] and Levine [14].) We now consider a blow-up solution $u(t)$ and note T its blow-up time. One can show that there is at least one blow-up point a (that is, $a \in \mathbb{R}^N$ such that $|u(a,t)| \to +\infty$ when $t \to T$). We will consider in this paper the case of a finite number of blow-up points (see [15]). More precisely, we will focus for simplicity on the case where there is only one blow-up point. We want to study the profile of the solution near blow-up, and the stability of such behavior with respect to initial data.

Standard tools, such as center manifold theory, have been proven nonefficient in this situation (cf. [6], [4]). In order to treat this problem, we introduce *similarity variables* (as in [10]):

$$y = \frac{x-a}{\sqrt{T-t}}, \tag{2}$$

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