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A SOLUTION TO QUANTUM KNIZHNIK-ZAMOLODCHIKOV EQUATIONS AND ITS APPLICATION TO EIGENVALUE PROBLEMS OF THE MACDONALD TYPE

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1. Introduction. I. Cherednik reconstructed the quantum Knizhnik-Zamolodchikov (QKZ) equations in the framework of the extended affine Weyl groups and the extended affine Hecke algebras for arbitrary root systems. Cherednik and S. Kato made a bridge between the QKZ equations and eigenvalue problems of the Macdonald type: a weighted symmetric sum of the solutions to the former gives rise to a solution of the latter [3], [4], [5], [9]. However, Cherednik and Kato's work on the QKZ equations is not devoted to obtaining concrete solutions. This is our goal. The main subjects of the present paper are (1) to give a solution to Cherednik's A_{n-1}-type QKZ equation through integral representations, especially using q-Selberg-type integrals, and (2) to apply this result to investigate eigenvalue problems of the Macdonald type. In Section 2, the definition of the QKZ equations is recalled. Next, we describe our main Theorem 3.1, after introducing the q-Selberg type integrals. This theorem is derived in Section 4 and Section 5. Subsequently, eigenvalue problems of the Macdonald type are considered in Section 6. We obtain a solution to such problems in Theorem 6.1. Continuation of this study leads to Conjecture 6.2, and we assert the following: integrals consisting of difference products are solutions of the eigenvalue problems of the Macdonald type. This implies that, if the conjecture holds, we can apply the (twisted) de Rham cohomology or homology theory to effectively investigate our eigenvalue problems. An integral representation of Macdonald symmetric polynomials, which is given in Section 7, provides evidence in support of our conjecture.

It is also noteworthy that our invariant measure with respect to the q-shift operation is realized by a continuous measure, not by Jackson's discrete measure. Almost all work on the q-difference equations of which we are aware focuses only on the Jackson measure. In contrast, at least from the viewpoint of orthogonal polynomials, the continuous measure is considered to be more fundamental and important. In this sense, the former studies are not sufficient. We believe that the present work should open the way to study in this area.

Before proceeding, we give a comment on related works. In [18], A. Varchenko and V. Tarasov study similar integrals given by the Jackson measure, in the context of the QKZ equations due to Frenkel-Reshetikhin [7], but their method

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