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ON THE MULTIPLICITIES OF THE DISCRETE SERIES OF SEMISIMPLE LIE GROUPS

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§0. Introduction. Let Γ be a torsion-free, cocompact, discrete subgroup of $G = SL_2(\mathbb{R})$. Then the dimension of the space $S_k(\Gamma)$ of modular forms of weight k is given by

$$(0.1)\qquad \begin{aligned} \dim S_k(\Gamma) &= \mathrm{Vol}(\Gamma\backslash G)\,\frac{k-1}{\sqrt{2}\,\pi} &&(k \geqq 3),\\ \dim S_2(\Gamma) &= \mathrm{Vol}(\Gamma\backslash G)\,\frac{1}{\sqrt{2}\,\pi}+1 &&(k=2). \end{aligned}$$

For our normalization of invariant measures, see §1. (We take $B(X, Y) = \mathrm{trace}(XY)/2$ as the bilinear form normalizing the measures in §1.) When $k = 2$, the correction term "+1" appears. This kind of correction term for lower weights is of our interest.

We can give a representation theoretic interpretation of the formula (0.1). For a locally compact topological group G and a cocompact discrete subgroup Γ of G, it is proved by I. M. Gel'fand, M. I. Graev, and I. I. Pyatetskii-Shapiro [GGP] that the representation r on $L^2(\Gamma\backslash G)$ by right translation splits into a discrete sum of a countable number of irreducible unitary representations, each of finite multiplicity m_π. Let D_k^+ $(k = 2, 3, \dots)$ be the holomorphic discrete series representation of $SL_2(\mathbb{R})$. By L. Clozel and P. Delorme [CD2], there exists a pseudocoefficient for any discrete series representation of a linear connected reductive group. Let φ be a pseudocoefficient of D_k^+. If $k \geqslant 3$, we have $\Theta_\pi(\varphi) = 0$ for any irreducible unitary representation π which is not equivalent to D_k^+, but if $k = 2$ we have $\Theta_{\mathrm{Trivial}}(\varphi) = -1$. It follows that the trace formula for φ is

$$(0.2)\qquad \begin{aligned} m_{D_k^+} &= \mathrm{Vol}(\Gamma\backslash G)\,\frac{k-1}{\sqrt{2}\,\pi} &&(k \geqq 3),\\ m_{D_2^+} &= m_{\mathrm{Trivial}} + \mathrm{Vol}(\Gamma\backslash G)\,\frac{1}{\sqrt{2}\,\pi} &&(k=2). \end{aligned}$$

Since the multiplicity m_{Trivial} of the trivial representation is "+1," we have (0.1). It turns out that the contribution of the trivial representation gives the correction

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