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ON STABLE MINIMAL SURFACES IN MANIFOLDS OF POSITIVE BI-RICCI CURVATURES

YING SHEN AND RUGANG YE

1. Introduction. In this paper we introduce a new version of curvature, which we call "bi-Ricci curvature," and present some estimates for the size of stable minimal hypersurfaces in 4- and 5-dimensional manifolds of positive bi-Ricci curvature, which include an estimate of the "homology radius" of the manifolds. We view these results as an initial step towards understanding (positive) bi-Ricci curvatures. There has been active research on various versions of curvatures: sectional curvatures, Ricci curvatures, scalar curvature, isotropy curvatures, etc. In particular, manifolds with positive curvatures have been the focus of attention. The central problem is to understand geometric and topological implications of positivity of various curvatures. In this regard, by the works of Schoen-Yau [ScY1], Gromov-Lawson [GL], and others, manifolds with positive scalar curvature have been well understood (though not yet completely understood). There has also been considerable progress on positive Ricci curvatures, but a global picture has not yet emerged.

The concept of bi-Ricci curvature lies somewhere between Ricci curvature and scalar curvature, and it may provide some kind of bridge between the two concepts. Of course, it deserves study for its own sake. The following is its precise definition.

Definition. Let M be a Riemannian manifold of dimension m and u, v be orthonormal tangent vectors. We set

$$BRc(u,v) = \mathrm{Ric}(u) + \mathrm{Ric}(v) - K(u,v),$$

and call it the *bi-Ricci curvature* in the directions u, v. Here $K(u,v)$ denotes the sectional curvature of the plane spanned by u, v.

Note that if $m = 3$, then $BRc = s/2$, where s denotes the scalar curvature. In general, $BR(u,v)$ is the sum of the sectional curvatures over all mutually orthogonal 2-planes containing u or v or both. It follows that the sum of BRc over an orthonormal base is ms. In particular, positive sectional curvature implies positive bi-Ricci curvature. The following example shows that the converse may not be true.

Example. In [Zi], W. Ziller considered a family of metrics g_s on $M = S^{2n+1} = SU(n+1)/SU(n)$. He showed that

$$\max K_M = \frac{(n+1)s^2}{2n}, \qquad \min K_M = 4 - 3s^2\,\frac{3(n+1)}{2n},$$

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