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ARITHMETIC GROUPS AND THE LENGTH SPECTRUM OF RIEMANN SURFACES

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1. Introduction. Arithmetic groups in the general context of Lie groups and algebraic groups were defined in the 1960s; see Borel and Harish-Chandra [1]. In the case of arithmetic Fuchsian groups (discrete subgroups of $PSL(2, \mathbf{R})$), Takeuchi [12] found a characterization in terms of the trace set which, for a Fuchsian group Γ, is defined as

$$Tr(\Gamma) = \{|tr(\gamma)|: \gamma \in \Gamma\}.$$

Takeuchi's characterization (see Section 3 below) is a number-theoretic one, but since it is related to the trace set, it contains a geometric meaning; namely, if $M = \mathbf{H}/\Gamma$ is the Riemann surface corresponding to a Fuchsian group Γ ($\mathbf{H}$ is the hyperbolic plane), then $Tr(\Gamma)$ can be defined as the set of the lengths of the closed geodesics of M. More precisely,

$$Tr(\Gamma) = Tr(M) := \{2\cosh(L(a)/2): a \text{ a closed geodesic of } M\} \cup \{2\},$$

where $L(a)$ stands for the length of a. This geometric meaning can be given more explicitly. I shall prove the following theorem.

THEOREM. *Let Γ be a cofinite Fuchsian group which contains at least one parabolic element. Then*

(i) *Γ is an arithmetic group if and only if there exists a finite constant C such that*

$$\#\{a \in Tr(\Gamma): a \leqslant n\} \leqslant 1 + Cu, \quad \forall n \geqslant 0.$$

(ii) *Γ is an arithmetic group derived from a quaternion algebra if and only if*

$$\mathrm{Gap}(\Gamma) := \inf\{|a - b|: a, b \in Tr(\Gamma), a \neq b\} > 0.$$

COROLLARY. *Let Γ and Γ' be two cofinite, noncompact Fuchsian groups. Let the trace sets $Tr(\Gamma) = \{a_1 < a_2 < a_3 < \cdots\}$ and $Tr(\Gamma') = \{a'_1 < a'_2 < a'_3 < \cdots\}$ both be listed in ascending order. Assume that Γ is arithmetic and Γ' is not arithmetic. Then there exists an integer $N = N(\Gamma, \Gamma')$, depending on the two groups, such that*

$$a'_i \leqslant a_i, \qquad \forall i \geqslant N.$$

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