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COMPOSITE DIFFERENTIABLE FUNCTIONS

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1. Introduction. In the early 1940s, Whitney proved that every $\mathscr{C}^\infty$ even function $f(x)$ can be written $f(x) = g(x^2)$, where g is $\mathscr{C}^\infty$ [28]. About twenty years later, Glaeser (answering a question posed by Thom in connection with the $\mathscr{C}^\infty$ preparation theorem) showed that a $\mathscr{C}^\infty$ function $f(x) = f(x_1, \dots, x_m)$ which is invariant under permutations of the coordinates can be expressed $f(x) = g(\sigma_1(x), \dots, \sigma_m(x))$, where g is $\mathscr{C}^\infty$ and the $\sigma_i(x)$ are the elementary symmetric polynomials [10]. Of course, not every $\mathscr{C}^\infty$ function $f(x) = f(x_1, \dots, x_m)$ which is constant on the fibres of a (proper or semiproper) real analytic mapping $y = \varphi(x)$, $y = (y_1, \dots, y_n)$, can be expressed as a composite $f = g \circ \varphi$, where g is $\mathscr{C}^\infty$. We will say that φ has the $\mathscr{C}^\infty$ *composite function property* if every $\mathscr{C}^\infty$ function $f(x)$ which is "formally a composite with φ" (see Definition 1.1 below) can be written $f = g \circ \varphi$, where $g(y)$ is $\mathscr{C}^\infty$. The theorem of Glaeser asserts that a semiproper real analytic mapping φ which is generically a submersion has the $\mathscr{C}^\infty$ composite function property. The $\mathscr{C}^\infty$ composite function property depends only on the image X of φ, which is a closed subanalytic set [1] (cf. Corollary 1.5 below). Bierstone and Milman have proved, more generally, that a closed "Nash subanalytic" set X has the $\mathscr{C}^\infty$ composite function property [1] (cf. [19], [23], [26]); the class of Nash subanalytic sets includes all semianalytic sets. The $\mathscr{C}^\infty$ composite function property is equivalent to several other natural geometric and algebraic conditions on a closed subanalytic set [6], in particular, to a formal semicoherence property (a stratified real version of the coherence theory of Oka and Cartan). Pawłucki has constructed an example of a closed subanalytic set which is not semicoherent [20]. Thus the $\mathscr{C}^\infty$ composite function property does not hold in general, but distinguishes an important class of subanalytic sets.

In this article, we introduce a new point of view towards Glaeser's theorem, with respect to which we can formulate a "$\mathscr{C}^k$ composite function property" that is satisfied by all semiproper real analytic mappings (Theorems 1.2 and 1.3 below). As a consequence, we see that a closed subanalytic set X satisfies the $\mathscr{C}^\infty$ composite function property if and only if the ring $\mathscr{C}^\infty(X)$ of $\mathscr{C}^\infty$ functions on X is the intersection of all finite differentiability classes (Corollary 1.5).

Let $k \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ denotes the nonnegative integers. Suppose that A

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