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# ON THE POSITIVITY OF THE CENTRAL CRITICAL VALUES OF AUTOMORPHIC $L$-FUNCTIONS FOR $GL(2)$

JIANDONG GUO

**1. Introduction.** The investigation of the nature of special values of $L$-functions occupies a central position in the study of automorphic forms, arithmetic, and geometry. Let $F$ be a number field and let $G$ be the group $GL(2)$ regarded as an algebraic group over $F$. Then to an automorphic cuspidal representation $\pi$ of $G$, we can attach an $L$-function $L(s, \pi)$ as in [JL], which satisfies the functional equation

$$L(s, \pi) = \varepsilon(s, \pi) L(1 - s, \tilde{\pi}),$$

where $\tilde{\pi}$ is the representation contragredient to $\pi$. In this case the properties concerning the nonvanishing and the algebraicity of the special values of $L(s, \pi)$ and its twists by finite idele class characters of $F$ have been studied extensively by many mathematicians (see, for example, [BFH1], [BFH2], [FH], [J1], [J2], [R], [S1], [S2], [S3], [V], [W1], [W2]). We now describe another problem in this direction.

We can decompose $L(s, \pi)$ as an Euler product over all places $v$ of $F$,

$$L(s, \pi) = \prod_v L(s, \pi_v)$$

if $\mathrm{Re}(s)$ is large enough [JL]. Here $\pi_v$ is a unitary representation of $G_v$ for each $v$. Suppose $\pi$ has trivial central character. Then $\pi$ is self-contragredient and we can assume that $\tilde{\pi} = \pi$. In this case we have $\overline{L(s, \pi_v)} = L(\bar{s}, \pi_v)$ for each $v$. This implies that $\overline{L(s, \pi)} = L(\bar{s}, \pi)$ if $\mathrm{Re}(s)$ is large enough. By continuity of $L(s, \pi)$, we have $L(a, \pi) = \overline{L(a, \pi)}$ for any real number $a$. In other words, $L(a, \pi)$ is real if $a$ is real. Here it is natural to ask about the sign of $L(a, \pi)$, in particular the sign of $L(1/2, \pi)$. For example, if $F = \mathbb{Q}$ and $\pi$ corresponds to a normalized newform on the upper half-plane of weight 2 with rational coefficients, then according to the Eichler-Shimura theory, there exists an elliptic curve $C$ over $\mathbb{Q}$ such that $L(s - 1/2, \pi) = L(s, C)$. Thus the value $L(1/2, \pi) = L(1, C)$ should be larger than or equal to zero by the Birch and Swinnerton-Dyer conjecture. Generally, we have that $L(a, \pi)$ is a continuous real function on the real axis. It is easy to check that if $a \geqslant 1/2$, then $L(a, \pi_v) > 0$ for all the places $v$ of $F$. By the results in [JS], the Euler product for $L(s, \pi)$ is absolutely convergent if $\mathrm{Re}(s) > 1$ and $L(s, \pi) \neq 0$ if $\mathrm{Re}(s) \geqslant 1$.

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