Vol. 83, No. 1 DUKE MATHEMATICAL JOURNAL © April 1996

CUSP FORMS OF WEIGHT 1 ASSOCIATED TO FERMAT CURVES

TONGHAI YANG

0. Introduction. Let Γ be a subgroup of $SL(2, \mathbb{Z})$ of finite index and k a positive integer. Let $M_k(\Gamma)$ and $S_k(\Gamma)$ be the spaces of modular forms and of cusp forms of weight k for Γ, respectively. Then modular forms of weight k for Γ can be viewed as sections of invertible sheaves (line bundles) on the corresponding modular curve $X(\Gamma)$; see for example [Mi]. Now the Riemann-Roch theorem gives explicit formulas for the dimensions of $M_k(\Gamma)$ and $S_k(\Gamma)$ when $k \geqslant 2$. For $k = 1$, however, the Riemann-Roch theorem does not provide an explicit formula. This is not surprising, because theorems of Weil, Jacquet-Langlands, and Deligne-Serre (see, for example, [Se] or [DS]) tells us that there is a one-to-one correspondence between cusp newforms for $\Gamma_0(N)$ of type $(1, \varepsilon)$ and two-dimensional irreducible Artin representations of conductor N and odd determinant ε satisfying the Artin conjecture. (By an *Artin representation* we mean a continuous finite-dimensional complex linear representation of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.) Langlands [La] and Tunnell [Tu] proved the Artin conjecture except for the type A_5. One would not expect a simple application of the Riemann-Roch theorem to give us deep results concerning Galois representations. In this note, however, we will give a very simple formula for the dimensions of the spaces of modular and of cusp forms of weight 1 for $\Phi(N)$, where $\Phi(N)$ is the subgroup of $SL(2, \mathbb{Z})$ associated to the Fermat curves defined as follows. Let Δ be the free subgroup of $\Gamma(2)$ on generators $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. One has $\Gamma(2) = \{\pm I\}\Delta$. Given a positive integer $N \geqslant 1$, the so-called Fermat group $\Phi(N)$ is defined as the subgroup of Δ generated by A^N, B^N, and the commutator $[\Delta, \Delta]$. The modular curve $X(\Phi(N))$ is isomorphic to the well-known Fermat curve F_N: $X^N + Y^N = Z^N$. In this article, we construct a canonical basis for the space of modular forms and cusp forms for $\Phi(N)$ of weight 1. In particular, one has the following.

THEOREM 1

$$\dim M_1(\Phi(N)) = \frac{([N/2] + 1)([N/2] + 2)}{2} \tag{1}$$

$$\dim S_1(\Phi(N)) = \frac{([(N-1)/2] - 1)[(N-1)/2]}{2}. \tag{2}$$

Received 2 May 1994. Revision received 23 August 1995.