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TENSOR PRODUCTS IN p-ADIC HODGE THEORY

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There is a classical relation between the p-adic absolute value of the eigenvalues of Frobenius on crystalline cohomology and Hodge numbers for a variety in characteristic p: "the Newton polygon lies on or above the Hodge polygon" [14], [1]. For a variety in characteristic p with a lift to characteristic 0, Fontaine conjectured, and Faltings proved, a more precise statement: There is an inequality which relates the slope of Frobenius on any Frobenius-invariant subspace of the crystalline cohomology to the Hodge filtration, restricted to that subspace [7], [4]. A vector space over a p-adic field together with a σ-linear endomorphism and a filtration which satisfies this inequality is called a weakly admissible filtered isocrystal (see Section 1 for the precise definition). The category of such objects is one possible p-adic analogue of the category of Hodge structures: in particular, it is an abelian category.

We give a new proof of Faltings's theorem (see [5]) that the tensor product of weakly admissible filtered isocrystals over a p-adic field is weakly admissible. By a similar argument, we also prove a characterization of weakly admissible filtered isocrystals with G-structure in terms of geometric invariant theory, which was conjectured by Rapoport and Zink [19]. Before Faltings, Laffaille [12] had proved the tensor product theorem in the case of filtered isocrystals over an unramified extension of $\mathbf{Q}_p$.

Faltings's proof works by reducing this problem of σ-linear algebra to a different problem of pure linear algebra, the problem of showing that the tensor product of two vector spaces, each equipped with a finite "semistable" set of filtrations, is semistable. The latter problem is solved by constructing suitable integral lattices (in [5]) or hermitian metrics (in [20]) on vector spaces with a semistable set of filtrations, just as one can prove that the tensor product of semistable bundles on an algebraic curve is semistable using Narasimhan-Seshadri's hermitian metrics [6], [16]. In this article, we can avoid the reduction from filtered isocrystals to filtered vector spaces.

The point is that Ramanan and Ramanathan's algebraic proof [17] that the tensor product of semistable vector bundles is semistable can be modified to apply directly to filtered isocrystals. We have an inequality to prove for a class of linear subspaces S of a tensor product $V \otimes W$. The inequality is obvious for sufficiently general subspaces S and also if S is a very special subspace, say if S is a decomposable subspace $S_1 \otimes S_2 \subset V \otimes W$. But it is not clear how to prove the

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