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MINIMUM HIGHER EIGENVALUES OF LAPLACIANS ON GRAPHS

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1. Introduction. Let $G = (V, E)$ be an undirected graph on n vertices (we allow multiple edges, self loops, and half-loops). The *Laplacian* of G, Δ, is the matrix $D - A$, in which A is G's adjacency matrix and D is the diagonal matrix whose (v, v) entry is the degree of v. If G is connected then it is well-known that Δ's eigenvalues, $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ satisfy $\mu_1 = 0$ and $\mu_2 > 0$.

In this article we will find (more or less explicitly) the smallest possible ith eigenvalue of Δ for a connected graph on n vertices; we will give examples of graphs achieving this value, and determine when there is a unique graph achieving this value.

The case $i \nmid n$ is the simplest to discuss. Examples of graphs achieving the smallest value are *stars*, which we define as follows.

Definition 1.1. By a *star of degree i* we mean a tree with one vertex (the *center*) of degree i and all other interior vertices of degree 2. Equivalently it is a graph consisting of i paths whose terminal vertices are all the same (all others being distinct). The *arms* of the star are the i paths; the *length* of the arm is the number of edges in it.

Figure 1 depicts a star of degree 3 with arms of length 4, 3, and 3.

THEOREM 1.2. *Let G be a connected graph on n vertices, and let $i \geqslant 2$ be an integer with $i \nmid n$. Then*

$$\mu_i(G) \geqslant 2 - 2\cos[\pi/(2m+1)],$$

where $m = \lfloor n/i \rfloor$. If $n \equiv 1 \pmod{i}$, then equality holds only for the star of degree i

A star of degree 3 A comb with 4 teeth

FIGURE 1. A star and a comb

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