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NILPOTENCE FOR MODULES OVER THE MOD 2 STEENROD ALGEBRA, I

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1. Introduction. In [4], Devinatz, Hopkins, and Smith proved the nilpotence theorem, a remarkable result which provides algebraic means for detecting nilpotence in the collection of homotopy classes of self-maps of any finite spectrum. This theorem has many important consequences, and so has opened up new approaches to studying homotopy theory (see [19], [10], [7], for examples). See [20] for a thorough discussion of this material; [5] also gives an overview of the nilpotence theorem and related results.

Let A be the mod 2 Steenrod algebra, and let M be a finite A-module. In this paper we show that there is an analogous result, Theorem 1.1, for detecting nilpotence in $\mathrm{Ext}_A^{**}(M, M)$. We hope that this leads to structure theorems for the category of finite A-modules, comparable to those for finite spectra in [8]. We begin to develop this material in a sequel [17]. Also, the nilpotence theorem for A-modules extends the strong parallel between results in stable homotopy theory and results for A-modules, as described in [11] and [16]. Some of these earlier A-module results have been used to prove results in homotopy theory via the Adams spectral sequence, as in [10] and [18]; we hope that one can do likewise with Theorem 1.1.

In order to state our main theorem, we need a few definitions. Given a Hopf algebra B over a field k of characteristic p, an *elementary* sub-Hopf algebra $E \subseteq B$ is a bicommutative Hopf algebra such that $e^p = 0$ for all $e \in IE$. (Here, IE denotes the augmentation ideal of E, $IE = \ker(\varepsilon\colon E \to k)$.) Of course, given any Hopf-algebra inclusion $\iota_C\colon C \to B$ and B-modules L and M, we have a restriction map $\iota_C^*\colon \mathrm{Ext}_B^{**}(L, M) \to \mathrm{Ext}_C^{**}(L, M)$. Note that if Γ is a (coassociative) B-coalgebra (so that there are B-module maps $\Delta\colon \Gamma \to \Gamma \otimes \Gamma$ and $\varepsilon\colon \Gamma \to k$ making certain diagrams commute), then $\mathrm{Ext}_B^{**}(\Gamma, k)$ is an (associative) algebra, via the map Δ^*. We prove the following result.

THEOREM 1.1. *Let B be a sub-Hopf algebra of the* mod 2 *Steenrod algebra A.*

(a) *Let Γ be a bounded below coassociative B-coalgebra; fix $z \in \mathrm{Ext}_B^{**}(\Gamma, \mathbf{F}_2)$. Then z is nilpotent if and only if $\iota_E^*(z)$ is nilpotent for all elementary sub-Hopf algebras E of B.*

(b) *Let M be a finite-dimensional B-module; fix $z \in \mathrm{Ext}_B^{**}(M, M)$. Then z is nilpotent under Yoneda composition if and only if $\iota_E^*(z)$ is nilpotent for all elementary sub-Hopf algebras E of B.*

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