

# ZEROS OF PRINCIPAL $L$ -FUNCTIONS AND RANDOM MATRIX THEORY

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**1. Introduction.** Our goal in this paper is to study the distribution of zeros of the Riemann zeta function as well as of more general  $L$ -functions.<sup>1</sup> According to conjectures of Langlands [14], the most general  $L$ -function is that attached to an automorphic representation of  $GL_N$  over a number field, and these in turn should be expressible as products of the “standard”  $L$ -functions  $L(s, \pi)$  attached to *cuspidal* automorphic representations of  $GL_m$  over the *rationals*. Such  $L$ -functions are therefore believed to be the building blocks for general  $L$ -functions, and we call them (principal) *primitive  $L$ -functions* of degree  $m$ . (They do not factor as products of such  $L$ -functions.)<sup>2</sup> For  $m = 1$  these are the Riemann zeta function  $\zeta(s)$  and Dirichlet  $L$ -functions  $L(s, \chi)$  with  $\chi$  primitive. For  $m = 2$  the analytic properties and functional equation of such  $L$ -functions were investigated by Hecke and Maass, and for  $m \geq 3$  by Godement and Jacquet [5]. We are interested in the fine structure of the distribution of the nontrivial zeros of such primitive  $L(s, \pi)$ . Let  $\rho^{(\pi)} = (1/2) + i\gamma^{(\pi)}$  denote these zeros. To motivate the formulation of our results, we begin by assuming the Riemann hypothesis (RH) for  $L(s, \pi)$ , that is, that  $\gamma^{(\pi)} \in \mathbf{R}$ . We order the  $\gamma^{(\pi)}$ 's (with multiplicities)

$$\cdots \leq \gamma_{-2}^{(\pi)} \leq \gamma_{-1}^{(\pi)} < 0 \leq \gamma_1^{(\pi)} \leq \gamma_2^{(\pi)} \cdots.$$

The number of  $\gamma$ 's in an interval  $[T, T + 1]$  is asymptotic to  $(m/2\pi) \log T$  as  $T \rightarrow \infty$  (see (2.11)). It follows that the numbers  $\tilde{\gamma}_j^{(\pi)} = (m/2\pi)\gamma_j \log |\gamma_j|$  have unit mean spacing. The problem is to understand the statistical nature of the sequence  $\tilde{\gamma}_j^{(\pi)}$ : Do they come down randomly (Poisson process) or do they follow a more revealing distribution?

In the case of the Riemann zeta function, following the original calculation by Montgomery [20] of the pair correlation (see below) and the extensive numerical calculations of Odlyzko [21], [22], it is now well accepted (but far from proven) that the consecutive spacings follow the Gaussian unitary ensemble (GUE) distribution from random matrix theory. That is, if  $\delta_n = \tilde{\gamma}_{n+1} - \tilde{\gamma}_n$  are the normalized

<sup>1</sup>The reader interested only in the Riemann zeta function  $\zeta(s)$  should read the paper with  $L(s, \pi)$  replaced by  $\zeta(s)$  and  $m = 1$  everywhere, in which case the results were announced in [26].

<sup>2</sup>It is quite plausible that these coincide with the primitive Dirichlet series introduced by Selberg [29] or the “arithmetic Dirichlet series” in Piatetski-Shapiro [24].

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