OPTIMAL SMOOTHING AND DECAY ESTIMATES FOR VISCOUSLY DAMPED CONSERVATION LAWS, WITH APPLICATIONS TO THE 2-D NAVIER-STOKES EQUATION

ERIC A. CARLEN AND MICHAEL LOSS

1. Introduction. A *viscously damped conservation law* is an equation of the form

$$\frac{\partial}{\partial t}u(t) = \Delta u(t) + \nabla\cdot(\mathbf{f}(x, u)u(x, t)) \tag{1.1}$$

where the vector field $\mathbf{f}$ in (1.1) has some functional dependence on u such that, whenever $\xi(x)$ is a smooth function of compact support,

$$\int_{\mathbb{R}^n} |\xi(x)|^q(\nabla\cdot\mathbf{f}(x, \xi))\,\mathrm{d}x = 0 \tag{1.2}$$

for all $q \geqslant 1$. In particular, if $u(\cdot, 0)$ is integrable, $(\mathrm{d}/\mathrm{d}t)\int u(x, t)\,\mathrm{d}x = 0$, and $\int u(x, t)\,\mathrm{d}x$ is a conserved quantity. The term involving the Laplacean represents the effects of some sort of "viscosity."

Useful and familiar examples are Burger's equation and the two-dimensional Navier-Stokes equations in the vorticity formulation. In one spatial dimension with $\mathbf{f}(x, u(\cdot)) = u(x)/2$, (1.1) becomes Burger's equation. (Note that (1.2) holds without $\mathbf{f}$ being divergence-free in this case.) In two spatial dimensions with

$$\mathbf{f}(x, \xi) = \int_{\mathbb{R}^2} S(x - y)\xi(y)\,\mathrm{d}^2y, \tag{1.3}$$

where

$$S(x) = \frac{1}{2\pi|x|^2}(-x_2, x_1), \tag{1.4}$$

(1.1) becomes a two-dimensional Navier-Stokes vorticity equation. These are the

Received 28 October 1994. Revision received 3 February 1995.

Carlen supported by National Science Foundation grant DMS-9207703.

Loss supported by National Science Foundation grant DMS-9207703 and the Erwin Schroedinger Institut in Vienna, Austria.