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MATRIX INTEGRALS, TODA SYMMETRIES, VIRASORO CONSTRAINTS, AND ORTHOGONAL POLYNOMIALS

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Symmetries of the infinite Toda lattice. The symmetries for the infinite Toda lattice,

$$\frac{\partial L}{\partial t_n} = \left[\frac{1}{2}(L^n)_s, L\right], \qquad n = 1, 2, \ldots, \tag{0.1}$$

viewed as isospectral deformations of bi-infinite tridiagonal matrices L, are time-dependent vector fields transversal to the Toda hierarchy; bracketing a symmetry with a Toda vector field yields another vector field in the hierarchy. As is well known, the Toda hierarchy is intimately related to the Lie algebra splitting of $\mathrm{gl}(\infty)$,

$$\mathrm{gl}(\infty) = \mathscr{D}_s \oplus \mathscr{D}_b \ni A = A_s + A_b, \tag{0.2}$$

into the algebras of skew-symmetric A_s and lower triangular (including the diagonal) matrices A_b (Borel matrices). We show that this splitting plays a prominent role also in the construction of the Toda symmetries and their action on τ-functions; it also plays a crucial role in obtaining the Virasoro constraints for matrix integrals, and it ties up elegantly with the theory of orthogonal polynomials.

Define matrices δ and ε, with $[\delta, \varepsilon] = 1$, acting on characters $\chi(z) = (\chi_n(z))_{n\in\mathbb{Z}} = (z^n)_{n\in\mathbb{Z}}$ as

$$\delta\chi = z\chi \quad \text{and} \quad \varepsilon\chi = \frac{\partial}{\partial z}\chi. \tag{0.3}$$

This enables us to define a wave operator S, a wave vector Ψ,

$$L = S\delta S^{-1} \quad \text{and} \quad \Psi = S\exp\left((1/2)\sum_1^\infty t_i z^i\right)\chi(z), \tag{0.4}$$

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