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ON THE DISTRIBUTION OF ZEROS OF LINEAR COMBINATIONS OF EULER PRODUCTS

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§1. Introduction. Very often in number theory, one needs to work with Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which can be viewed as elements of some natural $\mathbb{R}$-linear space $\mathscr{D}$. In many cases, it turns out that Euler products

$$L(s) = e^{i\omega} \prod_p \frac{1}{(1 - \alpha_{1p}p^{-s}) \cdots (1 - \alpha_{dp}p^{-s})} \tag{1.1}$$

play some type of preeminent role in the structure of $\mathscr{D}$, particularly if $\mathscr{D}$ is finite dimensional. On this score, we need only remind the reader of the important case, originating with Hecke, in which $\mathscr{D}$ is the Mellin transform of a space of holomorphic cusp forms of fixed weight and level. It is of course understood in (1.1) that $\alpha_{jp} \in \mathbb{C}$, $\omega \in \mathbb{R}$, $d \geqq 1$, and that p ranges over the rational primes. The factor $e^{i\omega}$ is slightly nonstandard and is introduced here solely for convenience in handling functional equations.

In the most interesting and arithmetically significant cases, the Euler products (1.1) also satisfy, or are conjectured to satisfy, an appropriate functional equation and an appropriate "Riemann Hypothesis", although on this latter point the situation is so mysterious that not a single example of validity (or, for that matter, failure) of a Riemann Hypothesis is known up to now.

At the same time, there are a whole host of arithmetically interesting Dirichlet series not possessing Euler products which satisfy a functional equation and which can often be expressed as linear combinations of Euler products of type (1.1). As illustrations it suffices to mention the zeta function attached to ideal classes in number fields and the Epstein zeta function attached to quadratic forms.

Our primary objective in this paper will be the study of zeros of general finite

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