Vol. 80, No. 3 DUKE MATHEMATICAL JOURNAL December 1995

SYMPLECTIC AND POISSON STRUCTURES OF CERTAIN MODULI SPACES, I

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Introduction. Let π be the fundamental group of a closed surface and G a Lie group, with an adjoint action invariant nondegenerate symmetric bilinear form on its Lie algebra g, not necessarily positive. For the case of a positive definite form and a compact connected G, Atiyah and Bott [2] constructed a symplectic structure on the smooth part of (the components of) the moduli space $\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)/G$ or, more generally, on twisted versions thereof, by symplectic reduction in infinite dimensions. Goldman [6] gave a finite-dimensional construction, but was unable to prove closedness without recourse to the infinite-dimensional picture. For general G and arbitrary nondegenerate form on g, the symplectic structure and, in particular, its closedness have recently been obtained by Weinstein by entirely finite-dimensional techniques [29] that extend an approach by Karshon [22]. Weinstein constructed a closed, equivariant 2-form on (the smooth part of) $\mathrm{Hom}(\pi, G)$ and showed by techniques from equivariant cohomology [1] that this 2-form descends to (the nonsingular part of) $\mathrm{Rep}(\pi, G)$. In this paper, we refine Weinstein's method by showing that in fact it yields (i) a symplectic structure on a certain smooth manifold $\mathscr{M}(\mathscr{P}, G)$ containing $\mathrm{Hom}(\pi, G)$ and, furthermore, (ii) a hamiltonian G-action on $\mathscr{M}(\mathscr{P}, G)$ preserving the symplectic structure, with momentum mapping μ from $\mathscr{M}(\mathscr{P}, G)$ to g^*, in such a way that the reduced space equals $\mathrm{Rep}(\pi, G)$. Our approach is somewhat more general in that it also applies to twisted moduli spaces (see Section 6). In particular, it yields the Narasimhan-Seshadri [27] moduli spaces of semistable holomorphic vector bundles by *symplectic reduction, applied to a smooth finite-dimensional symplectic manifold with a hamiltonian action of the unitary group*, which is *finite-dimensional.* Our result, apart from being interesting in its own right, reveals some interesting and attractive geometric properties of these twisted moduli spaces. First, it implies that, when the group G is compact, with reference to the decomposition into orbit types, such a twisted moduli space inherits a structure of *stratified symplectic space* in the sense of [28]. In particular, this yields a description of the behaviour of the symplectic or more generally Poisson structure of these moduli spaces in their singularities (in the appropriate sense); see our paper [13], where the Poisson geometry has been worked out explicitly in some special cases. Second, our result implies that the strata of these twisted moduli spaces have finite symplectic volume (see Section 7

Received 6 October 1994. Revision received 27 April 1995.
The author carried out this work in the framework of the VBAC research group of Europroj.