ON A CERTAIN NONARY CUBIC FORM AND RELATED EQUATIONS

R. C. VAUGHAN AND T. D. WOOLEY

For Professor Wolfgang Schmidt on the occasion of his 60th birthday

1. Introduction. In analytic number theory there are many situations in which, for certain exponents k_j satisfying $1 \leqslant k_1 < k_2 < \cdots < k_t$, one requires good estimates for the number, $S_s(P; \mathbf{k})$, of solutions to the system of equations

$$\sum_{i=1}^{s} \left(x_i^{k_j} - y_i^{k_j} \right) = 0 \quad (j = 1, \dots, t), \tag{1.1}$$

with $x_i, y_i \in [1, P] \cap \mathbb{Z}$. Thus, in the case $k_j = j$, estimates for $S_s(P; \mathbf{k})$, usually referred to as "Vinogradov's Mean Value Theorem", are central to the establishment of rather general estimates for exponential sums. These in turn lead to a number of theorems which, in the current state of knowledge, provide the best results available to us. For example, the best zero-free region for the Riemann zeta function is obtained in this way, as is the best upper bound, when the exponent k is large, for the smallest number $\tilde{G}(k)$ of variables for which the asymptotic formula holds in Waring's problem.

Besides their rôle in analytic number theory, estimates for the number of solutions of such systems as (1.1) provide useful insights into the distribution of rational points on certain algebraic varieties. For while the Hardy-Littlewood method will establish an asymptotic formula for the number of rational points, up to a given large height, lying on a variety satisfying suitable conditions (see Schmidt [29], and also Birch [3] for weaker results), such conditions are usually somewhat restrictive. In particular, the number of variables in the defining equations must be sufficiently large in terms of their degrees, and also sufficiently large in terms of the dimension of the singular locus. Thus in the present state of knowledge, the Hardy-Littlewood method fails, by a considerable margin, to resolve the conjectures of Manin et al. concerning the distribution of rational points on algebraic varieties, and in particular Fano varieties (see [1], [12], [27], and also [26, Chapter X, Conjecture 4.3]). Moreover, as will be apparent

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